Double Kostka polynomials and Hall bimodule

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Abstract. Double Kostka polynomials $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ are polynomials in t, indexed by double partitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$. As in the ordinary case, $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ is defined in terms of Schur functions $s_{\boldsymbol{\lambda}}(x)$ and Hall-Littlewood functions $P_{\boldsymbol{\mu}}(x;t)$. In this paper, we study combinatorial properties of $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ and $P_{\boldsymbol{\mu}}(x;t)$. In particular, we show that the Lascoux-Schützenberger type formula holds for $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ in the case where $\boldsymbol{\mu} = (-;\mu'')$. Moreover, we show that the Hall bimodule $\mathscr{M}$ introduced by Finkelberg-Ginzburg-Travkin is isomorphic to the ring of symmetric functions (with two types of variables) and the natural basis $\mathfrak{u}_{\boldsymbol{\lambda}}$ of $\mathscr{M}$ is sent to $P_{\boldsymbol{\lambda}}(x;t)$ (up to scalar) under this isomorphism. This gives an alternate approach for their result.

Introduction

Kostka polynomials $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$, indexed by double partitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$, were introduced in [S1, S2] as a generalization of ordinary Kostka polynomials $K_{\lambda,\mu}(t)$ indexed by partitions λ, μ. In this paper, we call them double Kostka polynomials. Let $\varLambda = \varLambda(y)$ be the ring of symmetric functions with respect to the variables $y = (y_1, y_2, \dots)$ over $\mathbf{Z}$. We regard $\varLambda \otimes \varLambda$ as the ring of symmetric functions $\varLambda(x^{(1)}, x^{(2)})$ with respect to two types of variables $x = (x^{(1)}, x^{(2)})$. Schur functions $\{s_{\boldsymbol{\lambda}}(x)\}$ gives a basis of $\varLambda \otimes \varLambda$. In [S1, S2], the function $P_{\boldsymbol{\mu}}(x;t)$ indexed by a double partition $\boldsymbol{\mu}$ was defined, as a generalization of the ordinary Hall-Littlewood function $P_\mu(x;t)$ indexed by a partition μ. $\{P_{\boldsymbol{\mu}}(x;t)\}$ gives a basis of $\mathbf{Z}[t] \otimes_{\mathbf{Z}} (\varLambda \otimes \varLambda)$, and as in the ordinary case, $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ is defined as the coefficient of the transition matrix between two basis $\{s_{\boldsymbol{\lambda}}(x)\}$ and $\{P_{\boldsymbol{\mu}}(x;t)\}$.

After the combinatorial introduction of $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ in [S1, S2], Achar-Henderson [AH] gave a geometric interpretation of double Kostka polynomials in terms of the intersection cohomology associated to the closure of orbits in the enhanced nilpotent cone, which is a natural generalization of the classical result of Lusztig [L1] that Kostka polynomials are interpreted by the intersection cohomology associated to the closure of nilpotent orbits in $\mathfrak{gl}_n$. At the same time, Finkelberg-Ginzburg-Travkin [FGT] studied the convolution algeba associated to the affine Grassmannian in connection with double Kostka polynomials and the geometry of the enhanced nilpotent cone. In particular, they introduced the Hall bimodule $\mathscr{M}$ (the mirabolic Hall bimodule in their terminology) as a generalization of the Hall algebra, and showed that $\mathscr{M}$ is isomorphic to $\varLambda \otimes \varLambda$ over $\mathbf{Z}[t, t^{-1}]$, and $P_{\boldsymbol{\lambda}}(x;t)$ is obtained as the image of the natural basis $\mathfrak{u}_{\boldsymbol{\lambda}}$ of $\mathscr{M}$.

In this paper, we study the combinatorial properties of $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ and $P_{\boldsymbol{\mu}}(x;t)$. In particular, we show that the Lascoux-Schützenberger type formula holds for $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$

in the case where $\boldsymbol{\mu} = (-;\mu'')$ (Theorem 3.12). Moreover, in Theorem 4.7, we give a more direct proof for the above mentioned result of [FGT] (in the sense that we don't appeal to the convolution algerba associated to the affine Grassmannian).

In the appendix, we give tables of double Kostaka polynomials for $2 \le n \le 5$, where n is the size of double partitions. The authors are grateful to J. Michel for the computer computation of those polynomials.

1. Double Kostka polynomials

1.1. First we recall basic properties of Hall-Littlewood functions and Kostka polynomials in the original setting, following [M]. Let $\varLambda = \varLambda(y) = \bigoplus_{n\ge 0}\varLambda^n$ be the ring of symmetric functions over $\mathbf{Z}$ with respect to the variables $y = (y_1, y_2, \dots)$, where $\varLambda^n$ denotes the free $\mathbf{Z}$-module of symmetirc functions of degree n. We put $\varLambda_{\mathbf{Q}} = \mathbf{Q}\otimes_{\mathbf{Z}}\varLambda$, $\varLambda^n_{\mathbf{Q}} = \mathbf{Q}\otimes_{\mathbf{Z}}\varLambda^n$. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, put $|\lambda| = \sum_{i=1}^k \lambda_i$. Let $\mathscr{P}_n$ be the set of partitions of n, i.e. the set of λ such that $|\lambda| = n$. Let s_λ be the Schur function associated to $\lambda \in \mathscr{P}_n$. Then $\{s_\lambda \mid \lambda \in \mathscr{P}_n\}$ gives a $\mathbf{Z}$-baisis of $\varLambda^n$. Let $p_\lambda \in \varLambda^n$ be the power sum symmetric funtion associated to λ. Then $\{p_\lambda \mid \lambda \in \mathscr{P}_n\}$ gives a $\mathbf{Q}$-basis of $\varLambda^n_{\mathbf{Q}}$. For $\lambda = (1^{m_1}, 2^{m_2}, \dots) \in \mathscr{P}_n$, define an integer z_λ by

$$z_\lambda = \prod_{i\ge 1} i^{m_i} m_i!. \tag{1.1.1}$$

Following [M, I], we introduce a scalar product on $\varLambda_{\mathbf{Q}}$ by $\langle p_\lambda, p_\mu\rangle = \delta_{\lambda\mu}z_\lambda$. It is known that $\{s_\lambda\}$ form an orthonormal basis of $\varLambda$.

1.2. Let $P_\lambda(y;t)$ be the Hall-Littlewood function associated to a partition λ. Then $\{P_\lambda \mid \lambda \in \mathscr{P}_n\}$ gives a $\mathbf{Z}[t]$-basis of $\varLambda^n[t] = \mathbf{Z}[t]\otimes_{\mathbf{Z}}\varLambda^n$, where t is an indeterminate. P_λ enjoys a property that

$$P_\lambda(y;0) = s_\lambda, \quad P_\lambda(y;1) = m_\lambda, \tag{1.2.1}$$

where $m_\lambda(y)$ is a monomial symmetric function associated to λ. Kostka polynomials $K_{\lambda,\mu}(t) \in \mathbf{Z}[t]$ $(\lambda,\mu \in \mathscr{P}_n)$ are defined by the formula

$$s_\lambda(y) = \sum_{\mu\in\mathcal{P}_n} K_{\lambda,\mu}(t)P_\mu(y;t). \tag{1.2.2}$$

Recall the dominance order $\lambda \le \mu$ in $\mathscr{P}_n$, which is defined by the condition $\lambda \le \mu$ if and only if $\sum_{j=1}^i \lambda_j \le \sum_{j=1}^i \mu_j$ for each $i \ge 1$. For each partition $\lambda = (\lambda_1, \dots, \lambda_k)$, we define an integer $n(\lambda)$ by $n(\lambda) = \sum_{i=1}^k (i-1)\lambda_i$. It is known that $K_{\lambda,\mu}(t) = 0$ unless $\lambda \ge \mu$, and that $K_{\lambda,\mu}(t)$ is a monic of degree $n(\mu) - n(\lambda)$ if $\lambda \ge \mu$ ([M, III, (6.5)]).

For $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathscr{P}_n$ with $\lambda_k > 0$, we define $z_\lambda(t) \in \mathbf{Q}(t)$ by

$$z_\lambda(t) = z_\lambda \prod_{i\ge 1}(1 - t^{\lambda_i})^{-1}, \tag{1.2.3}$$

where z_λ is as in (1.1.1). Following [M, III], we introduce a scalar product on $\varLambda_{\mathbf{Q}}(t) = \mathbf{Q}(t) \otimes_{\mathbf{Z}} \varLambda$ by $\langle p_\lambda, p_\mu \rangle = z_\lambda(t)\delta_{\lambda,\mu}$. Then $P_\lambda(y;t)$ form an orthogonal basis of $\varLambda[t] = \mathbf{Z}[t] \otimes_{\mathbf{Z}} \varLambda$. In fact, they are characterized by the following two properties ([M, III, (2.6) and (4.9)]);

$$P_\lambda(y;t) = s_\lambda(x) + \sum_{\mu<\lambda} w_{\lambda\mu}(t)s_\mu(x) \tag{1.2.4}$$

with $w_{\lambda\mu}(t) \in \mathbf{Z}[t]$, and

$$\langle P_\lambda, P_\mu \rangle = 0 \text{ unless } \lambda = \mu. \tag{1.2.5}$$

1.3. Let $\Xi = \Xi(x) = \varLambda(x^{(1)}) \otimes \varLambda(x^{(2)})$ be the ring of symmetric functions over $\mathbf{Z}$ with respect to variables $x = (x^{(1)}, x^{(2)})$, where $x^{(1)} = (x^{(1)}_1, x^{(1)}_2, \dots), x^{(2)} = (x^{(2)}_1, x^{(2)}_2, \dots)$. We denote it as $\Xi = \bigoplus_{n\ge 0} \Xi^n$, similarly to the case of $\varLambda$. Let $\mathscr{P}_{n,2}$ be the set of double partitions $\boldsymbol{\lambda} = (\lambda', \lambda'')$ such that $|\lambda'| + |\lambda''| = n$. For $\boldsymbol{\lambda} = (\lambda', \lambda'') \in \mathscr{P}_{n,2}$, we define a Schur function $s_{\boldsymbol{\lambda}}(x) \in \Xi^n$ by

$$s_{\boldsymbol{\lambda}}(x) = s_{\lambda'}(x^{(1)})s_{\lambda''}(x^{(2)}). \tag{1.3.1}$$

Then $\{s_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}_{n,2}\}$ gives a $\mathbf{Z}$-basis of Ξ^n. For an integer $r \ge 0$, put $p^{(1)}_r = p_r(x^{(1)}) + p_r(x^{(2)})$, and $p^{(2)}_r = p_r(x^{(1)}) - p_r(x^{(2)})$, where p_r is the power sum symmetric function in $\varLambda$. . For $\boldsymbol{\lambda} \in \mathscr{P}_{n,2}$, we define $p_{\boldsymbol{\lambda}}(x) \in \Xi^n$ by

$$p_{\boldsymbol{\lambda}} = \prod_i p^{(1)}_{\lambda'_i} \prod_j p^{(2)}_{\lambda''_j}, \tag{1.3.2}$$

where $\boldsymbol{\lambda} = (\lambda', \lambda'')$ such that $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{k'})$, $\lambda'' = (\lambda''_1, \lambda''_2, \dots, \lambda''_{k''})$ with $\lambda'_{k'}, \lambda''_{k''} > 0$. Then $\{p_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}_{n,2}\}$ gives a $\mathbf{Q}$-basis of $\Xi^n_{\mathbf{Q}}$. For $\boldsymbol{\lambda} \in \mathscr{P}_{n,2}$, we define functions $z^{(1)}_{\boldsymbol{\lambda}}(t), z^{(2)}_{\boldsymbol{\lambda}}(t) \in \mathbf{Q}(t)$ by

$$z^{(1)}_{\boldsymbol{\lambda}}(t) = \prod_{j=1}^{k'} (1 - t^{\lambda'_j})^{-1}, \qquad z^{(2)}_{\boldsymbol{\lambda}}(t) = \prod_{j=1}^{k''} (1 + t^{\lambda''_j})^{-1}. \tag{1.3.3}$$

For $\boldsymbol{\lambda} \in \mathscr{P}_{n,2}$, we define an integer $z_{\boldsymbol{\lambda}}$ by $z_{\boldsymbol{\lambda}} = 2^{k'+k''} z_{\lambda'} z_{\lambda''}$. We now define a function $z_{\boldsymbol{\lambda}}(t) \in \mathbf{Q}(t)$ by

$$z_{\boldsymbol{\lambda}}(t) = z_{\boldsymbol{\lambda}} z^{(1)}_{\boldsymbol{\lambda}}(t) z^{(2)}_{\boldsymbol{\lambda}}(t). \tag{1.3.4}$$

Let $\Xi[t] = \mathbf{Z}[t] \otimes_{\mathbf{Z}} \Xi$ be the free $\mathbf{Z}[t]$-module, and $\Xi_{\mathbf{Q}}(t) = \mathbf{Q}(t) \otimes_{\mathbf{Z}} \Xi$ be the $\mathbf{Q}(t)$-space. Then $\{p_{\boldsymbol{\lambda}}(x) \mid \boldsymbol{\lambda} \in \mathscr{P}_{n,2}\}$ gives a basis of $\Xi^n_{\mathbf{Q}}(t)$. We define a scalar product on $\Xi_{\mathbf{Q}}(t)$ by

$$\langle p_{\boldsymbol{\lambda}}, p_{\boldsymbol{\mu}} \rangle = \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}} z_{\boldsymbol{\lambda}}(t).$$

We express a double partition $\boldsymbol{\lambda} = (\lambda', \lambda'')$ as $\lambda' = (\lambda'_1, \dots, \lambda'_k), \lambda'' = (\lambda''_1, \dots, \lambda''_k)$ with some k, by allowing zero on parts λ'_i, λ''_i. We define a composition $c(\boldsymbol{\lambda})$ of n by

$$c(\boldsymbol{\lambda}) = (\lambda'_1, \lambda''_1, \lambda'_2, \lambda''_2, \dots, \lambda'_k, \lambda''_k).$$

We define a partial order $\boldsymbol{\lambda} \ge \boldsymbol{\mu}$ on $\mathscr{P}_{n,2}$ by the the condition $c(\boldsymbol{\lambda}) \ge c(\boldsymbol{\mu})$, where $\ge$ is the dominance order on the set of compositions of n defined in a similar way as in the case of partitions.

The following fact is known.

Proposition 1.4 ([S1, S2])**.** *There exists a unique function $P_{\boldsymbol{\lambda}}(x;t) \in \Xi_{\mathbf{Q}}[t]$ satisfying the following properties.*

(i) *$P_{\boldsymbol{\lambda}}$ is expressed as a linear combination of Schur functions $s_{\boldsymbol{\mu}}$ as*

$$P_{\boldsymbol{\lambda}}(x;t) = s_{\boldsymbol{\lambda}}(x) + \sum_{\boldsymbol{\mu} < \boldsymbol{\lambda}} u_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) s_{\boldsymbol{\mu}}(x)$$

with $u_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbf{Q}(t)$.

(ii) *$\langle P_{\boldsymbol{\lambda}}, P_{\boldsymbol{\mu}} \rangle = 0$ unless $\boldsymbol{\lambda} = \boldsymbol{\mu}$.*

Remark 1.5. $P_{\boldsymbol{\lambda}}$ is called the Hall-Littlewood function associated to a double partiton $\boldsymbol{\lambda}$. More generally, Hall-Littlewood functions associated to r-partitions of n was introduced in [S1]. However the arguments in [S1] is based on a fixed total order which is compatible with the partial order $\ge$ on $\mathscr{P}_{n,2}$ even in the case of double partitions. In [S2, Theorem 2.8], the closed formula for $P_{\boldsymbol{\lambda}}$ is given in the case of double partitions. This implies that $P_{\boldsymbol{\lambda}}$ is independent of the choice of the total order, and is deterimned uniquely as in the above proposition. (The uniqueness of $P_{\boldsymbol{\lambda}}$ also follows from the result of Achar-Henderson, see Theorem 2.4.)

1.6. By Proposition 1.4, $\{P_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}_{n,2}\}$ gives a basis of $\Xi_{\mathbf{Q}}(t)$. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{n,2}$, we define a function $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbf{Q}(t)$ by the formula

$$s_{\boldsymbol{\lambda}}(x) = \sum_{\boldsymbol{\mu} \in \mathcal{P}_{n,2}} K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) P_{\boldsymbol{\mu}}(x;t).$$

$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ are called the Kostka functions associated to double partitions. For each $\boldsymbol{\lambda} = (\lambda', \lambda'') \in \mathscr{P}_{n,2}$, put $n(\boldsymbol{\lambda}) = n(\lambda' + \lambda'') = n(\lambda') + n(\lambda'')$. We define an integer $a(\boldsymbol{\lambda})$ by

$$a(\boldsymbol{\lambda}) = 2n(\boldsymbol{\lambda}) + |\lambda''|. \tag{1.6.1}$$

The following result was proved in [S2, Prop. 3.3].

Proposition 1.7. *$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbf{Z}[t]$. $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = 0$ unless $\boldsymbol{\lambda} \ge \boldsymbol{\mu}$. If $\boldsymbol{\lambda} \ge \boldsymbol{\mu}$, $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ is a monic of degree $a(\boldsymbol{\mu}) - a(\boldsymbol{\lambda})$, hence $K_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(t) = 1$. In particular, $P_{\boldsymbol{\lambda}}(x;t) \in \Xi^n[t]$, and $u_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbf{Z}[t]$.*

1.8. Since $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ is a polynomial in t associated to double partitions, we call it the double Kostka polynomial. Put $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{a(\boldsymbol{\mu})} K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t^{-1})$. By Proposition 1.7,

$\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ is again contained in $\mathbf{Z}[t]$, which we call the modified double Kostka polynomial. In the case of Kostka polynomial $K_{\lambda,\mu}(t)$, we also put $\widetilde{K}_{\lambda,\mu}(t) = t^{n(\mu)}K_{\lambda,\mu}(t^{-1})$. By 1.2, $\widetilde{K}_{\lambda,\mu}(t)$ is a polynomial in $\mathbf{Z}[t]$, which is called the modified Kostka polynomial.

Following [S1, S2], we give a combinatorial characterization of $\widetilde{K}_{\lambda,\mu}(t)$ and $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$. In order to discuss both cases simultaneously, we introduce some notation. For $r = 1, 2$, put $W_{n,r} = S_n \ltimes (\mathbf{Z}/2\mathbf{Z})^n$. Hence $W_{n,r}$ is the symmetric group S_n of degree n if $r = 1$, and is the Weyl group of type C_n if $r = 2$. For a (not necessarily irreducible) character χ of $W_{n,r}$, we define the fake degree $R(\chi)$ by

$$R(\chi) = \frac{\prod_{i=1}^n (t^{ir} - 1)}{|W_{n,r}|} \sum_{w \in W_{n,r}} \frac{\varepsilon(w)\chi(w)}{\det_{V_0}(t - w)}, \tag{1.8.1}$$

where ε is the sign character of $W_{n,r}$, and V_0 is the reflection representation of $W_{n,r}$ if $r = 2$ (i.e., $\dim V_0 = n$), and its restriction on S_n if $r = 1$. Let $R(W_{n,r}) = \bigoplus_{i=1}^N R_i$ be the coinvariant algebra over $\mathbf{Q}$ associated to $W_{n,r}$, where N is the number of positive roots of the root system of type C_n (resp. type A_{n-1}) if $r = 2$ (resp. $r = 1$). Then $R(W_{n,r})$ is a graded $W_{n,r}$-module, and we have

$$R(\chi) = \sum_{i=1}^N \langle \chi, R_i \rangle_{W_{n,r}} t^i, \tag{1.8.2}$$

where $\langle\, , \rangle_{W_{n,r}}$ is the inner product of characters of $W_{n,r}$. It follows that $R(\chi) \in \mathbf{Z}[t]$. It is known that irreducible characters of $W_{n,r}$ are parametrized by $\mathscr{P}_{n,r}$ (we use the convention that $\mathscr{P}_{n,1} = \mathscr{P}_n$). We denote by $\chi^{\boldsymbol{\lambda}}$ the irreducible character of $W_{n,r}$ corresponding to $\boldsymbol{\lambda} \in \mathscr{P}_{n,r}$. (Here we use the parametrization such that the identity character corresponds to $\boldsymbol{\lambda} = ((n), -)$ if $r = 2$, and $\lambda = (n)$ if $r = 1$.) We define a square matrix $\Omega = (\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}})_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ by

$$\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}} = t^N R(\chi^{\boldsymbol{\lambda}} \otimes \chi^{\boldsymbol{\mu}} \otimes \varepsilon). \tag{1.8.3}$$

We have the following result. Note that Theorem 5.4 in [S1] is stated for a fixed total order on $\mathcal{P}_{n,2}$. But in our case, it can be replaced by the partial order (see Remark 1.5).

Proposition 1.9 ([S1, Thm. 5.4]). *Assume that $r = 2$. There exist unique matrices $P = (p_{\boldsymbol{\lambda},\boldsymbol{\mu}})$, $\varLambda = (\xi_{\boldsymbol{\lambda},\boldsymbol{\mu}})$ over $\mathbf{Q}[t]$ satisfying the equation*

$$P\varLambda\, {}^tP = \Omega,$$

subject to the condition that $\varLambda$ is a diagonal matrix and that

$$p_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \begin{cases} 0 & \textit{unless } \boldsymbol{\mu} \le \boldsymbol{\lambda}\ , \\ t^{a(\boldsymbol{\lambda})} & \textit{if } \boldsymbol{\lambda} = \boldsymbol{\mu}\ . \end{cases}$$

Then the entry $p_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ of the matrix P coincides with $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$.

A simialr result holds for the case $r = 1$ by replacing $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{n,2}$ by $\lambda, \mu \in \mathscr{P}_n$, and by replacing $a(\boldsymbol{\lambda})$ by $n(\lambda)$.

1.10. Assume that $\boldsymbol{\lambda} = (-, \lambda'') \in \mathscr{P}_{n,2}$. If $\boldsymbol{\mu} \le \boldsymbol{\lambda}$, then $\boldsymbol{\mu}$ is of the form $\boldsymbol{\mu} = (-, \mu'')$ with $\mu'' \le \lambda''$. Thus $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = 0$ unless $\boldsymbol{\mu}$ satisfies this condition. The following result was showm by Achar-Henderson [AH] by a geometric method (see Proposition 2.5 (ii)). We give below an alternate proof based on Proposition 1.9.

Proposition 1.11. *Assume that $\boldsymbol{\lambda} = (-, \lambda''), \boldsymbol{\mu} = (-, \mu'') \in \mathscr{P}_{n,2}$. Then*

$$\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^n \widetilde{K}_{\lambda'',\mu''}(t^2). \tag{1.11.1}$$

In particular, we have

$$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = K_{\lambda'',\mu''}(t^2). \tag{1.11.2}$$

Proof. (1.11.2) follows from (1.11.1). We show (1.11.1). We shall compute $\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}} = t^N R(\chi^{\boldsymbol{\lambda}} \otimes \chi^{\boldsymbol{\mu}} \otimes \varepsilon)$ for $\boldsymbol{\lambda} = (-, \lambda''), \boldsymbol{\mu} = (-.\mu'')$. $\chi^{\boldsymbol{\lambda}}$ corresponds to the irreducible representation of S_n with character $\chi^{\lambda''}$, extended by the action of $(\mathbf{Z}/2\mathbf{Z})^n$ such that any factor $\mathbf{Z}/2\mathbf{Z}$ acts non-trivially. This is the same for $\chi^{\boldsymbol{\mu}}$. Hence $\chi^{\boldsymbol{\lambda}} \otimes \chi^{\boldsymbol{\mu}}$ corresponds to the representation of S_n with character $\chi^{\lambda''} \otimes \chi^{\mu''}$, extended by the trivial action of $(\mathbf{Z}/2\mathbf{Z})^n$. Thus $\chi^{\boldsymbol{\lambda}} \otimes \chi^{\boldsymbol{\mu}} \otimes \varepsilon$ corresponds to the representation of S_n with character $\chi^{\lambda''} \otimes \chi^{\mu''} \otimes \varepsilon'$, extended by the action of $(\mathbf{Z}/2\mathbf{Z})^n$ such that any factor $\mathbf{Z}/2\mathbf{Z}$ acts non-trivially, where ε' denote the sign character of S_n. Let $\{s_1, \dots, s_n\}$ be the set of simple reflections of W_n. We identify the symmetric algebra $S(V_0^*)$ of V_0 with the polynomial ring $\mathbf{R}[y_1, \dots, y_n]$ with the natural W_n action, where s_i permutes y_i and y_{i+1} $(1 \le i \le n-1)$, and s_n maps y_n to $-y_n$. Then $(\mathbf{Z}/2\mathbf{Z})^n$-invariant subalgebra of $\mathbf{R}[y_1, \dots, y_n]$ coincides with $\mathbf{R}[y_1^2, \dots, y_n^2]$. It follows that the $(\mathbf{Z}/2\mathbf{Z})^n$-invariant subalgerba $R(W_n)^{(\mathbf{Z}/2\mathbf{Z})^n}$ of $R(W_n)$ is isomorphic to $R(S_n)$ as graded algebras, where the degree $2i$-part of $R(W_n)^{(\mathbf{Z}/2\mathbf{Z})^n}$ corresponds to the degree i part of $R(S_n)$. Let X be the subspace of $R(W_n)$ consisitng of vectors on which $(\mathbf{Z}/2\mathbf{Z})^n$ acts in such a way that each factor $\mathbf{Z}/2\mathbf{Z}$ acts non-trivially. Then $X = y_1 \cdots y_n R(W_n)^{(\mathbf{Z}/2\mathbf{Z})^n}$. It follows that

$$R(\chi^{\boldsymbol{\lambda}} \otimes \chi^{\boldsymbol{\mu}} \otimes \varepsilon)(t) = t^n R(\chi^{\lambda''} \otimes \chi^{\mu''} \otimes \varepsilon')(t^2).$$

Since $N = n^2$ for W_n-case, and $N = n(n-1)/2$ for S_n-case, this implies that

$$\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{2n} \omega_{\lambda'',\mu''}(t^2) \tag{1.11.3}$$

We consider the embedding $\mathscr{P}_n \hookrightarrow \mathscr{P}_{n,2}$ by $\lambda'' \mapsto (-, \lambda'')$. This embedding is compatible with the partial order of $\mathscr{P}_n$ and $\mathscr{P}_{n,2}$, and in fact, $\mathcal{P}_n$ is identifed with the subset $\{\boldsymbol{\mu} \in \mathcal{P}_{n,2} \mid \boldsymbol{\mu} \le (-, (n))\}$ of $\mathscr{P}_{n,2}$. We consider the matrix equation $P\varLambda\, {}^tP = \Omega$ as in Proposition 1.9 for $r = 2$. Let $P_0, \varLambda_0, \Omega_0$ be the submatrices of $P, \varLambda, \Omega$ obtained by restricting the indices from $\mathscr{P}_{n,2}$ to $\mathscr{P}_n$. Then these matrices satisfy the relation $P_0 \varLambda_0\, {}^tP_0 = \Omega_0$. By (1.11.3) Ω_0 coincides with $t^{2n}\Omega'(t^2)$, where

Ω' denotes the matrix Ω in the case $r=1$. If we put $P' = t^{-n}P_0, \Lambda' = \Lambda_0$, we have a matrix equation $P'\Lambda'\,{}^tP' = \Omega'(t^2)$. Note that the (λ'',λ'')-entry of P' coincides with $t^{-n}t^{a(\boldsymbol{\lambda})} = t^{2n(\lambda'')}$. Hence P', Λ', Ω' satisfy all the requirements in Proposition 1.9 for the case $r=1$, by replacing t by t^2. Now by Proposition 1.9, we have $t^{-n}\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = \widetilde{K}_{\lambda'',\mu''}(t^2)$ as asserted. □

As a corollary, we have

Corollary 1.12. *Assume that* $\boldsymbol{\lambda} = (-,\lambda'')$. *Then* $P_{\boldsymbol{\lambda}}(x;t) = P_{\lambda''}(x^{(2)};t^2)$.

Proof. Since $\boldsymbol{\lambda} = (-,\lambda'')$, we have $s_{\boldsymbol{\lambda}}(x) = s_{\lambda''}(x^{(2)})$. By (1.11.2), we have

$$s_{\lambda''}(x^{(2)}) = \sum_{\mu''\in\mathcal{P}_n} K_{\lambda'',\mu''}(t^2)P_{\boldsymbol{\mu}}(x;t).$$

We have also

$$s_{\lambda''}(x^{(2)}) = \sum_{\mu''\in\mathcal{P}_n} K_{\lambda'',\mu''}(t^2)P_{\mu''}(x^{(2)};t^2).$$

Since $(K_{\lambda'',\mu''}(t^2))$ is a non-singular matrix indexed by $\mathcal{P}_n$, the assertion follows. □

2. Geometric interpretation of double Kostka polynomials

2.1. In [L1], Lusztig gave a geometric interpretation of Kostka polynomials in terms of the intersection cohomology complex associated to the nilpotent orbits of $\mathfrak{gl}_n$. Let V be an n-dimensional vector space over an algebraically closed field k, and put $G = GL(V)$. Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{g}_{\mathrm{nil}}$ the nilpotent cone of $\mathfrak{g}$. G acts on $\mathfrak{g}_{\mathrm{nil}}$ by the adjoint action, and the set of G-orbits in $\mathfrak{g}_{\mathrm{nil}}$ is in bijective correspondence with $\mathscr{P}_n$ via the Jordan normal form of nilpotent elements. We denote by $\mathscr{O}_\lambda$ the G-orbit corresponding to $\lambda\in\mathcal{P}_n$. Let $\overline{\mathscr{O}}_\lambda$ be the closure of $\mathscr{O}_\lambda$ in $\mathfrak{g}_{\mathrm{nil}}$. Then we have $\overline{\mathscr{O}}_\lambda = \coprod_{\mu\le\lambda}\mathscr{O}_\mu$, where $\mu\le\lambda$ is the dominance order of $\mathscr{P}_n$. Let $A_\lambda = \mathrm{IC}(\overline{\mathscr{O}}_\lambda, \bar{\mathbf{Q}}_l)$ be the intersection cohomology complex of $\bar{\mathbf{Q}}_l$-sheaves, and $\mathscr{H}^i_xA_\lambda$ the stalk at $x\in\overline{\mathscr{O}}_\lambda$ of the i-th cohomology sheaf $\mathscr{H}^iA_\lambda$. Lusztig's result is stated as follows.

Theorem 2.2 ([L1, Thm. 2])**.** $\mathscr{H}^iA_\lambda = 0$ *for odd* i. *For each* $x\in\mathscr{O}_\mu\subset\overline{\mathscr{O}}_\lambda$,

$$\widetilde{K}_{\lambda,\mu}(t) = t^{n(\lambda)}\sum_{i\ge0}(\dim\mathscr{H}^{2i}_xA_\lambda)t^i.$$

2.3. The geometric interpretation of double Kostka polynomials analogous to Theorem 2.2 was established by Achar-Henderson [AH]. We follow the setting in 2.1. Consider the direct product $\mathscr{X} = \mathfrak{g}\times V$, on which G acts as $g:(x,v)\mapsto(gx,gv)$, where gv is the natural action of G on V. Put $\mathscr{X}_{\mathrm{nil}} = \mathfrak{g}_{\mathrm{nil}}\times V$. $\mathscr{X}_{\mathrm{nil}}$ is a G-stable subset of $\mathscr{X}$, and is called the enhanced nilpotent cone. It is known by Achar-Henderson [AH] and by Travkin [T] that the set of G-orbits in $\mathscr{X}_{\mathrm{nil}}$ is in bijective correspondence with $\mathscr{P}_{n,2}$. The correspondence is given as follows; take

$(x,v) \in \mathscr{X}_{\mathrm{nil}}$. Put $E^x = \{g \in \mathrm{End}(V) \mid gx = xg\}$. Then $W = E^x v$ is an x-stable subspace of V. Let λ' be the Jordan type of $x|_W$, and λ'' the Jordan type of $x|_{V/W}$. Then $\boldsymbol{\lambda} = (\lambda', \lambda'') \in \mathscr{P}_{n,2}$, and the assignment $(x,v) \mapsto \boldsymbol{\lambda}$ gives the required correspondence. We denote by $\mathscr{O}_{\boldsymbol{\lambda}}$ the G-orbit corresponding to $\boldsymbol{\lambda} \in \mathscr{P}_{n,2}$. The closure relation for $\mathscr{O}_{\boldsymbol{\lambda}}$ was described by [AH, Thm. 3.9] as follows;

$$\overline{\mathscr{O}}_{\boldsymbol{\lambda}} = \coprod_{\boldsymbol{\mu} \le \boldsymbol{\lambda}} \mathscr{O}_{\boldsymbol{\mu}}, \tag{2.3.1}$$

where the partial order $\boldsymbol{\mu} \le \boldsymbol{\lambda}$ is the one defined in 1.3. We consider the intersection cohomology complex $A_{\boldsymbol{\lambda}} = \mathrm{IC}(\overline{\mathscr{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l)$ on $\mathscr{X}_{\mathrm{nil}}$ associated to $\boldsymbol{\lambda} \in \mathscr{P}_{n,2}$. The following result was proved by Achar-Henderson.

Theorem 2.4 ([AH, Thm. 5.2])**.** *Assume that $A_{\boldsymbol{\lambda}}$ is attached to the enhanced nilpotent cone. Then $\mathscr{H}^i A_{\boldsymbol{\lambda}} = 0$ for odd i. For $z \in \mathscr{O}_{\boldsymbol{\mu}} \subset \overline{\mathscr{O}}_{\boldsymbol{\lambda}}$,*

$$\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{a(\boldsymbol{\lambda})} \sum_{i \ge 0} (\dim \mathscr{H}^{2i}_z A_{\boldsymbol{\lambda}}) t^{2i}.$$

Note that $\mathscr{H}^{2i}$ corresponds to t^{2i} in the enhanced case, which is different from the correspondence $\mathscr{H}^{2i} \leftrightarrow t^i$ in the $\mathfrak{g}_{\mathrm{nil}}$ case. As a corollary, we have

Proposition 2.5 ([AH, Cor. 5.3])**.**

(i) *$\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbf{Z}_{\ge 0}[t]$. Moreover, only powers of t congruent to $a(\boldsymbol{\lambda})$ modulo 2 occur in the polynomial.*

(ii) *Assume that $\boldsymbol{\lambda} = (-, \lambda'')$, $\boldsymbol{\mu} = (-, \mu'')$. Then $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^n \widetilde{K}_{\lambda'',\mu''}(t^2)$.*

(iii) *Assume that $\boldsymbol{\lambda} = (\lambda', -)$ and $\boldsymbol{\mu} = (\mu', \mu'')$. Then $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = \widetilde{K}_{\lambda',\mu'+\mu''}(t^2)$.*

Proof. For the sake of completeness, we give the proof here. (i) is clear from the theorem. For (ii), take $\boldsymbol{\lambda} = (-, \lambda'')$. Then by the correspondence given in 2.3, if $(x,v) \in \mathscr{O}_{\boldsymbol{\lambda}}$, then $v = 0$, and $x \in \mathscr{O}_{\lambda''}$. It follows that $\mathscr{O}_{\boldsymbol{\lambda}} = \mathscr{O}_{\lambda''}$ and that $A_{\boldsymbol{\lambda}} \simeq A_{\lambda''}$. $z \in \mathscr{O}_{\boldsymbol{\mu}}$ is also wtitten as $z = (x, 0)$ with $x \in \mathscr{O}_{\mu''}$. Then (ii) follows by comparing Theorem 2.2 and Theorem 2.4. For (iii), it was proved in [AH, Lemma 3.1] that $\overline{\mathscr{O}}_{\boldsymbol{\lambda}} = \overline{\mathscr{O}}_{\lambda'} \times V$ for $\boldsymbol{\lambda} = (\lambda', -)$. Thus $\mathrm{IC}(\overline{\mathscr{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l) \simeq \mathrm{IC}(\overline{\mathscr{O}}_{\lambda'}, \bar{\mathbf{Q}}_l) \boxtimes (\bar{\mathbf{Q}}_l)_V$, where $(\bar{\mathbf{Q}}_l)_V$ is the constant sheaf $\bar{\mathbf{Q}}_l$ on V. It follows that $\mathscr{H}^{2i}_z A_{\boldsymbol{\lambda}} = \mathscr{H}^{2i}_x A_{\lambda'}$ for $z = (x,v) \in \mathscr{O}_{\boldsymbol{\mu}}$. Since $x \in \mathscr{O}_{\mu'+\mu''}$, (iii) follows from Theorem 2.2 (note that $a(\boldsymbol{\lambda}) = 2n(\lambda')$). □

Remark 2.6. Proposition 2.5 (ii) was also proved in Proposition 1.11 by a combinatorial method. We don't know whether (iii) is proved in a combinatorial way. However if we admit that $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ depends only on $\mu' + \mu''$ for $\boldsymbol{\lambda} = (\lambda', -)$ (this is a consequence of (iii)), a similar argument as in the proof of Proposition 1.1 can be applied.

Proposition 2.5 (iii) implies the following.

Corollary 2.7. *For $\nu \in \mathscr{P}_n$, we have*

$$P_\nu(x^{(1)}; t^2) = \sum_{\nu = \mu' + \mu''} t^{|\mu''|} P_{(\mu',\mu'')}(x; t).$$

Proof. It follows from Proposition 2.5 (iii) that $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{|\mu''|}K_{\lambda',\mu'+\mu''}(t^2)$ for $\boldsymbol{\lambda} = (\lambda', -)$. Since $s_{\boldsymbol{\lambda}}(x) = s_{\lambda'}(x^{(1)})$, we have

$$\begin{aligned} s_{\lambda'}(x^{(1)}) &= \sum_{\boldsymbol{\mu}\in\mathcal{P}_{n,2}} K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)P_{\boldsymbol{\mu}}(x;t) \\ &= \sum_{\boldsymbol{\mu}\in\mathcal{P}_{n,2}} K_{\lambda',\mu'+\mu''}(t^2)t^{|\mu''|}P_{\boldsymbol{\mu}}(x;t) \\ &= \sum_{\nu\in\mathcal{P}_n} K_{\lambda',\nu}(t^2) \sum_{\nu=\mu'+\mu''} t^{|\mu''|}P_{(\mu',\mu'')}(x;t). \end{aligned}$$

On the other hand, we have

$$s_{\lambda'}(x^{(1)}) = \sum_{\nu\in\mathcal{P}_n} K_{\lambda',\nu}(t^2)P_\nu(x^{(1)};t^2).$$

Since $(K_{\lambda',\nu}(t^2))$ is a non-singular matrix, we obtain the required formula. □

Remark 2.8. The formula in Corollary 2.7 suggests that the behaviour of $P_{\boldsymbol{\mu}}(x;t)$ at $t = 1$ is different from that of ordinally Hall-Littlewood functions given in (1.2.1). In fact, by Corollary 1.12, $P_{(-,\nu)}(x;t) = P_\nu(x^{(2)};t^2)$. Hence $P_{(-,\nu)}(x;1) = m_\nu(x^{(2)})$ by (1.2.1). Also by (1.2.1) $P_\nu(x^{(1)};1) = m_\nu(x^{(1)})$. Then by Corollary 2.7, we have

$$m_\nu(x^{(1)}) = m_\nu(x^{(2)}) + \sum_{\nu=\mu'+\mu'',\mu'\neq\emptyset} P_{(\mu',\mu'')}(x;1).$$

This formula shows that a certain cancelation occurs in the expression of $P_{\boldsymbol{\mu}}(x;1)$ as a sum of monomials. Concerning this, we will have a related result later in Proposition 3.23.

2.9. There exists a geometric realization of double Kostka polynomials in terms of the exotic nilpotent cone instead of the enhanced nilpotent cone. Let V be a $2n$-dimensional vector space over an algebraically closed field k of odd characteristic. Let $G = GL(V)$ and θ an involutive automorphism of G such that $G^\theta = Sp(V)$. Put $H = G^\theta$. Let $\mathfrak{g}$ be the Lie algebra of G. θ induces a linear automorphism of order 2 on $\mathfrak{g}$, which we denote also by θ. $\mathfrak{g}$ is decomposed as $\mathfrak{g} = \mathfrak{g}^\theta \oplus \mathfrak{g}^{-\theta}$, where $\mathfrak{g}^{\pm\theta}$ is the eigenspace of θ with eigenvalue ± 1. Thus $\mathfrak{g}^{\pm\theta}$ are H-invariant subspaces in $\mathfrak{g}$. We consider the direct product $\mathscr{X} = \mathfrak{g}^{-\theta} \times V$, on which H acts diagonally. Put $\mathfrak{g}^{-\theta}_{\mathrm{nil}} = \mathfrak{g}^{-\theta} \cap \mathfrak{g}_{\mathrm{nil}}$. Then $\mathfrak{g}^{-\theta}_{\mathrm{nil}}$ is H-stable, and we consider $\mathscr{X}_{\mathrm{nil}} = \mathfrak{g}^{-\theta}_{\mathrm{nil}} \times V$. $\mathscr{X}_{\mathrm{nil}}$ is an H-invariant subset of $\mathscr{X}$, and is called the exotic nilpotent cone. It is known by Kato [K1] that the set of H-orbits in $\mathscr{X}_{\mathrm{nil}}$ is in bijective correspondence with $\mathscr{P}_{n,2}$. We denote by $\mathscr{O}_{\boldsymbol{\lambda}}$ the H-orbit corresponding to $\boldsymbol{\lambda} \in \mathscr{P}_{n,2}$. It is also known by [AH] that the closure relations for $\mathscr{O}_{\boldsymbol{\lambda}}$ are given by the partial order $\le$ on $\mathscr{P}_{n,2}$. We consider the intersection cohomology complex $A_{\boldsymbol{\lambda}} = \mathrm{IC}(\overline{\mathscr{O}}_{\boldsymbol{\lambda}}, \bar{\mathbf{Q}}_l)$ on $\mathscr{X}_{\mathrm{nil}}$. The following result was proved by Kato [K2], and [SS2], independently.

Theorem 2.10. *Assume that $A_{\boldsymbol\lambda}$ is attached to the exotic nilpotent cone. Then $\mathscr{H}^i A_{\boldsymbol\lambda} = 0$ unless $i \equiv 0 \pmod 4$. For $z \in \mathscr{O}_{\boldsymbol\mu} \subset \overline{\mathscr{O}}_{\boldsymbol\lambda}$, we have*

$$\widetilde{K}_{\boldsymbol\lambda,\boldsymbol\mu}(t) = t^{a(\boldsymbol\lambda)} \sum_{i\ge 0} (\dim \mathscr{H}^{4i}_z A_{\boldsymbol\lambda}) t^{2i}.$$

2.11. Let W_n be the Weyl group of type C_n. The advantage of the use of the exotic nilpotent cone relies on the fact that it has a good relationship with representations of Weyl groups, as explained below. Let B be a θ-stable Borel subgroup of G. Then B^θ is a Borel subgrouop of H, and we denote by $\mathscr{B}$ the flag variety H/B^θ of H. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_n \subset V$ be the (full) isotropic flag fixed by B^θ. Hence M_n is a maximal isotropic subspace of V. Put

$$\widetilde{\mathscr{X}}_{\mathrm{nil}} = \{(x, v, gB^\theta) \in \mathfrak{g}^{-\theta}_{\mathrm{nil}} \times V \times \mathscr{B} \mid g^{-1}x \in \operatorname{Lie} B, g^{-1}v \in M_n\},$$

and define a map $\pi_1 : \widetilde{\mathscr{X}}_{\mathrm{nil}} \to \mathscr{X}_{\mathrm{nil}}$ by $(x, v, gB^\theta) \mapsto (x, v)$. Then $\widetilde{\mathscr{X}}_{\mathrm{nil}}$ is smooth, irreducible and π_1 is proper surjective. Let $V_{\boldsymbol\lambda}$ be the irreducible representation of W_n corresponidng to $\chi^{\boldsymbol\lambda}$ ($\boldsymbol\lambda \in \mathscr{P}_{n,2}$). We consider the direct iamge $(\pi_1)_*\bar{\mathbf{Q}}_l$ of the constant sheaf $\bar{\mathbf{Q}}_l$ on $\widetilde{\mathscr{X}}_{\mathrm{nil}}$. The following result is an analogue of the Springer correspondene for reductive groups, and was proved by Kato [K1], and [SS1], independently.

Theorem 2.12. $(\pi_1)_*\bar{\mathbf{Q}}_l[\dim \mathscr{X}_{\mathrm{nil}}]$ *is a semisimple perverse sheaf on $\mathscr{X}_{\mathrm{nil}}$, equipped with W_n-action, and is decomposed as*

$$(\pi_1)_*\bar{\mathbf{Q}}_l[\dim \mathscr{X}_{\mathrm{nil}}] \simeq \bigoplus_{\boldsymbol\lambda \in \mathscr{P}_{n,2}} V_{\boldsymbol\lambda} \otimes A_{\boldsymbol\lambda}[\dim \mathscr{O}_{\boldsymbol\lambda}], \tag{2.12.1}$$

where $A_{\boldsymbol\lambda}[\dim \mathscr{O}_{\boldsymbol\lambda}]$ is a simple perverse sheaf on $\mathscr{X}_{\mathrm{nil}}$.

2.13. For each $z = (x, v) \in \mathscr{X}_{\mathrm{nil}}$, put

$$\mathscr{B}_z = \{gB^\theta \in \mathscr{B} \mid g^{-1}x \in \operatorname{Lie} B, g^{-1}v \in M_n\}.$$

$\mathscr{B}_z$ is isomorphic to $\pi_1^{-1}(z)$, and is called the Springer fibre. Since $\mathscr{H}^i_z((\pi_1)_*\bar{\mathbf{Q}}_l) \simeq H^i(\mathscr{B}_z, \bar{\mathbf{Q}}_l)$, $H^i(\mathscr{B}_z, \bar{\mathbf{Q}}_l)$ has a structure of W_n-module, which we call the Springer representation of W_n. Put $K = (\pi_1)_*\bar{\mathbf{Q}}_l$. By taking the stalk at $z \in \mathscr{X}_{\mathrm{nil}}$ of the i-th cohomology of both sides in (2.12.1), we have an isomorphism of W_n-modules,

$$\mathscr{H}^i_z K \simeq H^i(\mathscr{B}_z, \bar{\mathbf{Q}}_l) \simeq \bigoplus_{\boldsymbol\lambda \in \mathscr{P}_{n,2}} V_{\boldsymbol\lambda} \otimes \mathscr{H}^{i+\dim \mathscr{O}_{\boldsymbol\lambda} - \dim \mathscr{X}_{\mathrm{nil}}}_z A_{\boldsymbol\lambda}.$$

Since $\dim \mathscr{X}_{\mathrm{nil}} - \dim \mathscr{O}_{\boldsymbol\lambda} = 2a(\boldsymbol\lambda)$ (see [SS2, (5.7.1)], this together with Theorem 2.10 imply the following result.

Proposition 2.14. *Assume that $z \in \mathscr{O}_{\boldsymbol{\mu}}$. Then $H^i(\mathscr{B}_z, \bar{\mathbf{Q}}_l) = 0$ for odd i, and we have*

$$\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = \sum_{i \ge 0} \langle H^{2i}(\mathscr{B}_z, \bar{\mathbf{Q}}_l), V_{\boldsymbol{\lambda}} \rangle_{W_n} t^i,$$

namely, the coefficient of t^i in $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ is given by the multiplicity of $V_{\boldsymbol{\lambda}}$ in the W_n-module $H^{2i}(\mathscr{B}_z, \bar{\mathbf{Q}}_l)$.

3. Combinatorial properties of $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ and $P_{\boldsymbol{\mu}}(x;t)$

3.1. In [AH], Achar-Henderson gave a formula expressing double Kostka polynomials in terms of various ordinary Kostka polynomials. We consider the enhanced nilpotent cone $\mathscr{X}_{\mathrm{nil}} = \mathfrak{g}_{\mathrm{nil}} \times V$ as in 2.3, under the assumption that k is an algebraic closure of a finite field $\mathbf{F}_q$. Take $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}_{n,2}$. For each $z = (x, v) \in \mathscr{O}_{\boldsymbol{\mu}}$ and $\boldsymbol{\nu} = (\nu', \nu'')$, we define a variety $\mathscr{G}^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}$ by

$$\begin{aligned} \mathscr{G}^{\boldsymbol{\mu}}_{\boldsymbol{\nu}} = \{W \subset V \mid W &: x\text{-stable subspace}, v \in W, \\ &x|_W \text{ type} : \nu',\ x|_{V/W} \text{ type} : \nu'' \}. \end{aligned} \tag{3.1.1}$$

Note that if $z \in \mathscr{O}_{\boldsymbol{\mu}}(\mathbf{F}_q)$, the variety $\mathscr{G}^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}$ is defined over $\mathbf{F}_q$, and one can count the cardinality $|\mathscr{G}^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(\mathbf{F}_q)|$ of $\mathbf{F}_q$-fixed points in $\mathscr{G}^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}$. Clearly, $|\mathscr{G}^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(\mathbf{F}_q)|$ is independent of the choice of $z \in \mathscr{O}_{\boldsymbol{\mu}}(\mathbf{F}_q)$.

Proposition 3.2 (Achar-Henderson [AH, Prop. 5.8])**.** *Let $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}_{n,2}$.*

(i) *There exists a polynomial $g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t) \in \mathbf{Z}[t]$ such that $|\mathscr{G}_{z,\boldsymbol{\nu}}(\mathbf{F}_q)| = g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(q)$ for any finite field $\mathbf{F}_q$ such that $z \in \mathscr{O}_{\boldsymbol{\mu}}(\mathbf{F}_q)$.*

(ii) *Take $\boldsymbol{\lambda} = (\lambda', \lambda''), \boldsymbol{\nu} = (\nu', \nu'')$. Then we have*

$$\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{a(\boldsymbol{\lambda}) - 2n(\boldsymbol{\lambda})} \sum_{\substack{\nu' \le \lambda' \\ \nu'' \le \lambda''}} g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t^2) \widetilde{K}_{\lambda'\nu'}(t^2) \widetilde{K}_{\lambda''\nu''}(t^2). \tag{3.2.1}$$

3.3. The formula (3.2.1) can be rewritten as

$$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{|\mu''| - |\lambda''|} \sum_{\boldsymbol{\nu} = (\nu', \nu'') \in \mathscr{P}_{n,2}} t^{2n(\boldsymbol{\mu}) - 2n(\boldsymbol{\nu})} g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t^{-2}) K_{\lambda'\nu'}(t^2) K_{\lambda''\nu''}(t^2). \tag{3.3.1}$$

Note that $g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t)$ is a generalization of Hall polynomials. If $\boldsymbol{\mu} = (-, \mu'')$, then $z = (x, v)$ with $v = 0$. In that case, $g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t)$ coincides with the original Hall polynomial $g^{\mu''}_{\nu'\nu''}(t)$ given in [M, II, 4]. In particular, if $g^{\mu}_{\nu'\nu''}(t) \ne 0$, then $g^{\mu}_{\nu'\nu''}(t)$ is a polynomial with degree $n(\mu) - n(\nu') - n(\nu'')$ and leading coefficient $c^{\mu}_{\nu'\nu''}$, where $c^{\mu}_{\nu'\nu''}$ is the Littlewood-Richardson coeffcient determined by the following conditions; for

partitions λ, μ, ν,

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda. \tag{3.3.2}$$

For partitions λ, μ, ν, we define a polynomial $f^\lambda_{\mu\nu}(t)$ by

$$P_\mu(y;t)P_\nu(y;t) = \sum_\lambda f^\lambda_{\mu\nu}(t)P_\lambda(y;t). \tag{3.3.3}$$

Then it is known by [M, III, (3.6)] that

$$g^\lambda_{\mu\nu}(t) = t^{n(\lambda)-n(\mu)-n(\nu)} f^\lambda_{\mu\nu}(t^{-1}). \tag{3.3.4}$$

We now assume that $\boldsymbol{\mu} = (-, \mu'')$. Substituting (3.3.4) into (3.3.1), we have

$$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{|\lambda'|} \sum_{\nu',\nu''} f^{\mu''}_{\nu'\nu''}(t^2)K_{\lambda'\nu'}(t^2)K_{\lambda''\nu''}(t^2). \tag{3.3.5}$$

Lemma 3.4. *Assume that* $\boldsymbol{\mu} = (-, \mu'')$. *Then we have*

$$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{|\lambda'|} \sum_{\nu',\nu''} f^{\mu''}_{\nu'\nu''}(t^2)K_{\lambda'\nu'}(t^2)K_{\lambda''\nu''}(t^2), \tag{3.4.1}$$

$$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^{|\lambda'|} \sum_{\eta} c^\eta_{\lambda'\lambda''}K_{\eta,\mu''}(t^2). \tag{3.4.2}$$

Proof. The first equality is given in (3.3.5). We show the second equality. One can write

$$\begin{aligned} s_{\lambda'}(y) &= \sum_{\nu'} K_{\lambda'\nu'}(t)P_{\nu'}(y;t), \\ s_{\lambda''}(y) &= \sum_{\nu''} K_{\lambda'',\nu''}(t)P_{\nu''}(y;t). \end{aligned}$$

Hence

$$\begin{aligned} s_{\lambda'}(y)s_{\lambda''}(y) &= \sum_{\nu',\nu''} K_{\lambda'\nu'}(t)K_{\lambda''\nu''}(t)P_{\nu'}(y;t)P_{\nu''}(y;t) \\ &= \sum_{\nu',\nu''}\sum_{\mu''} f^{\mu''}_{\nu'\nu''}(t)K_{\lambda'\nu'}(t)K_{\lambda''\nu''}(t)P_{\mu''}(y;t). \end{aligned} \tag{3.4.3}$$

On the other hand,

$$
\begin{aligned}
s_{\lambda'}(y)s_{\lambda''}(y) &= \sum_\eta c^\eta_{\lambda'\lambda''}s_\eta(y) \\
&= \sum_\eta c^\eta_{\lambda'\lambda''}\sum_{\mu''}K_{\eta,\mu''}(t)P_{\mu''}(y;t).
\end{aligned}
\tag{3.4.4}
$$

By comparing (3.4.3) and (3.4.4), we have, for each λ', λ'' and μ'',

$$
\sum_\eta c^\eta_{\lambda'\lambda''}K_{\eta,\mu''}(t) = \sum_{\nu',\nu''} f^{\mu''}_{\nu'\nu''}(t)K_{\lambda'\nu'}(t)K_{\lambda''\nu''}(t).
$$

This proves the second equality. □

3.5. For $\lambda, \mu \in \mathscr{P}_n$, let $SST(\lambda, \mu)$ be the set of semistandard tableaux of shape λ and weight μ. For a semistandard tableau S, the charge $c(S)$ is defined as in [M, III, 6]. Then Lascoux-Schützenberger theorem ([M, III, (6.5)]) asserts that

$$
K_{\lambda\mu}(t) = \sum_{S\in SST(\lambda;\mu)} t^{c(S)}. \tag{3.5.1}
$$

In what follows, we shall prove an analogue of (3.5.1) for double Kostka polynomials $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ for some special cases. Let $\boldsymbol{\lambda} = (\lambda', \lambda'') \in \mathscr{P}_{n,2}$. A pair $T = (T_+, T_-)$ is called a semistandard tableau of shape $\boldsymbol{\lambda}$ if T_+ (resp. T_-) is a semistandard tableau of shape λ' (resp. λ'') with respect to the letters $1, \dots, n$. We denote by $SST(\lambda)$ the set of semistandard tableaux of shape $\boldsymbol{\lambda}$. $T \in SST(\boldsymbol{\lambda})$ is regarded as a usual semistandard tableau associated to a skew diagram; write $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{k'})$ with $\lambda'_{k'} > 0$, and $\lambda'' = (\lambda''_1, \lambda''_2, \dots, \lambda''_{k''})$ with $\lambda''_{k''} > 0$. Put $a = \lambda''_1$. We define a partition $\xi = (\xi_1, \dots, \xi_{k'+k''}) \in \mathscr{P}_{n+ak'}$ by

$$
\xi_i = \begin{cases} \lambda'_i + a & \text{for } 1 \le i \le k' , \\ \lambda''_{i-k'} & \text{for } k'+1 \le i \le k'+k''. \end{cases}
$$

We define a partition $\theta = (a^{k'})$ of rectangular shape. Then $\theta \subset \xi$, and the skew diagram $\xi - \theta$ consist of connected component of shape λ' and λ''. Thus $T \in SST(\boldsymbol{\lambda})$ can be identified with a semistandard tableau $\widetilde{T}$ of shape $\xi - \theta$. Assume $\pi \in \mathscr{P}_n$. We say that $T \in SST(\lambda)$ has weight π if the corresponding tableau $\widetilde{T}$ has shape $\xi - \theta$ and weight π. We denote by $SST(\boldsymbol{\lambda}, \pi)$ the set of semistandard tableau of shape $\boldsymbol{\lambda}$ and weight π.

The set $SST(\boldsymbol{\lambda}, \pi)$ is described as follows; for a partition $\nu \in \mathscr{P}_m$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}^n_{\ge 0}$ such that $|\alpha| = \sum_i \alpha_i = m$, let $SST(\nu; \alpha)$ be the set of semistandard tableau of shape ν and weight α. Then we have

$$SST(\boldsymbol{\lambda},\pi) = \coprod_{\substack{\alpha+\beta=\pi \\ |\alpha|=|\lambda'|}} (SST(\lambda',\alpha)\times SST(\lambda'',\beta)). \tag{3.5.2}$$

Remark 3.6. Usually, the weight of a semistandard tableau is assumed to be a partition. Here we need to consider the weight which is not a partition. But this gives no essential difference. In fact, we consider the set $SST(\nu;\alpha)$. S_n acts on $\mathbf{Z}^n_{\ge 0}$ by a permutation of factors. We denote by $O(\alpha)$ the S_n-orbit of α in $\mathbf{Z}^n_{\ge 0}$. There exists a unique $\mu \in O(\alpha)$ such that μ is a partition. Then we have $|SST(\nu;\alpha)| = |SST(\nu;\mu)|$. This follows from (5.12) in [M, I] and the discussion below (though it is not written explicitly).

3.7. For (an ordinary) semistandard tableau S, a word $w(S)$ is defined as a sequence of letters $1,\dots,n$, reading from right to left, and top to down. This definition works for the semistandard tableau assoicaited to a skew-diagram. For a semistandard tableau $T=(T_+,T_-)\in SST(\boldsymbol{\lambda})$, we define the associated word $w(T)$ by $w(T)=w(T_+)w(T_-)$. Hence $w(T)$ coinicdes with $w(\widetilde{T})$.

Following [M, I, 9], we introduce a notion of lattice permutation. A word $w = a_1\cdots a_N$ consisting of letters $1,\dots,n$ is called a lattice permutation if for $1\le r\le N$ and $1\le i\le n-1$, the number of occurrences of the letter i in $a_1\cdots a_r$ is $\ge$ the number of occurrences of the letter $i+1$. We denote by $SST^0(\boldsymbol{\lambda},\pi)$ the set of semistandard tableau $T\in SST(\boldsymbol{\lambda},\pi)$ such that $w(T)$ is a lattice permutation.

Lemma 3.8. *Assume that $\boldsymbol{\lambda}\in\mathscr{P}_{n,2}$, $\pi\in\mathscr{P}_n$. There exists a bijective map*

$$\varTheta : SST(\boldsymbol{\lambda},\pi) \xrightarrow{\sim} \coprod_{\nu\in\mathscr{P}_n} (SST^0(\boldsymbol{\lambda},\nu)\times SST(\nu,\pi)) \tag{3.8.1}$$

Proof. Under the correspondence $T\leftrightarrow\widetilde{T}$ in 3.5, the set $SST(\boldsymbol{\lambda},\pi)$ can be identified with the set $SST(\xi-\theta,\pi)$. Then (3.8.1) is a special case of the bijection given in [M, I, (9.4)]. In (9.4), this bijection is explicitly constructed. □

Corollary 3.9. *Assume that $\boldsymbol{\lambda}=(\lambda',\lambda'')\in\mathscr{P}_{n,2}$, $\nu\in\mathscr{P}_n$. Then we have*

$$|SST^0(\boldsymbol{\lambda},\nu)| = c^{\nu}_{\lambda',\lambda''}.$$

Proof. We prove the corollary by modifying the discussion in [M, I, 9]. By [M, I, (5.12)], we have

$$s_{\lambda'}(y) = \sum_{S'\in SST(\lambda')} y^{S'},$$
$$s_{\lambda''}(y) = \sum_{S''\in SST(\lambda'')} y^{S''}.$$

It follows that

$$s_{\lambda'}(y)s_{\lambda''}(y) = \sum_{T \in SST(\boldsymbol{\lambda})} y^T.$$

By a similar argument as in the proof of (5.14) in [M, I], we have

$$|SST(\boldsymbol{\lambda}, \pi)| = \langle s_{\lambda'}s_{\lambda''}, h_\pi \rangle,$$

where h_π is a complete symmetric function assoicated to π. Similarly, we have $|SST(\nu, \pi)| = \langle s_\nu, h_\pi \rangle$. Then by (3.8.1), we have

$$\langle s_{\lambda'}s_{\lambda''}, h_\pi \rangle = \sum_{\nu \in \mathscr{P}_n} |SST^0(\boldsymbol{\lambda}, \nu)| \langle s_\nu, h_\pi \rangle$$

for any $\pi \in \mathscr{P}_n$. It follows that

$$s_{\lambda'}s_{\lambda''} = \sum_{\nu \in \mathscr{P}_n} |SST^0(\boldsymbol{\lambda}, \nu)| s_\nu. \tag{3.9.1}$$

On the other hand, by (3.3.2) we have

$$s_{\lambda'}s_{\lambda''} = \sum_{\nu \in \mathscr{P}_n} c^\nu_{\lambda', \lambda''} s_\nu. \tag{3.9.2}$$

By comparing the coefficient of s_ν in (3.9.1) with (3.9.2), we obtain the result. $\square$

Remark 3.10. The Littlewood-Richardson rule is a combinatorial procedure of computing Littlewood-Richardson coefficients. In [M, I, (9.2)] it is stated that $c^\nu_{\lambda', \lambda''}$ is equal to the number of smsitandard tableaux T of shape $\nu - \lambda'$ and weight λ'' such that $w(T)$ is a lattice permutation. Hence Corollary 3.9 gives a variant of the Littlewood-Richardson rule.

3.11. Assume that $\boldsymbol{\lambda} \in \mathscr{P}_{n,2}$ and $\mu'' \in \mathscr{P}_n$. For $T \in SST(\boldsymbol{\lambda}, \mu'')$, write $\Theta(T) = (D, S)$, with $S \in SST(\nu, \mu'')$ for some ν. We define a charge $c(T)$ of T by $c(T) = c(S)$, where $c(S)$ is the charge of S as in (3.5.1). The following formula is an analogue of Lascoux-Schützenberger thorem for the double Kostka polynomial $K_{\boldsymbol{\lambda}, \boldsymbol{\mu}}(t)$ in the case where $\boldsymbol{\mu} = (-, \mu'')$.

Theorem 3.12. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_{n,2}$, and assume that $\boldsymbol{\mu} = (-, \mu'')$. Then*

$$K_{\boldsymbol{\lambda}, \boldsymbol{\mu}}(t) = t^{|\lambda'|} \sum_{T \in SST(\boldsymbol{\lambda}, \mu'')} t^{2c(T)}.$$

Proof. We define a map $\Psi : SST(\boldsymbol{\lambda}, \mu'') \to \coprod_{\nu \in \mathscr{P}_n} SST(\nu, \mu'')$ by $T \mapsto S$, where $\Theta(T) = (D, S)$. Then by Corollary 3.9, for each $S \in SST(\nu, \mu'')$, the set $\Psi^{-1}(S)$ has the cardinality $c^\nu_{\lambda' \lambda''}$, and, by definition, any element $T \in \Psi^{-1}(S)$ has the charge $c(T) = c(S)$. Hence

$$
\begin{aligned}
\sum_{T\in SST(\boldsymbol{\lambda},\mu'')} t^{c(T)} &= \sum_{\nu\in\mathscr{P}_n}\sum_{S\in SST(\nu,\mu'')} c^{\nu}_{\lambda'\lambda''}t^{c(S)} \\
&= \sum_{\nu\in\mathscr{P}_n} c^{\nu}_{\lambda'\lambda''}K_{\nu,\mu''}(t)
\end{aligned}
$$

since $K_{\nu,\mu''}(t)=\sum_S t^{c(S)}$ by (3.5.1). Now the theorem follows from (3.4.2). $\square$

Corollary 3.13. *Assume that $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_{n,2}$ with $\boldsymbol{\mu}=(-,\mu'')$. Then we have*

$$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(1)=|SST(\boldsymbol{\lambda},\mu'')|.$$

3.14. Here we recall the explicit construction of $\chi^{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda}=(\lambda',\lambda'')\in\mathscr{P}_{n,2}$. Put $|\lambda'|=m', |\lambda''|=m''$. Let $\chi^{\lambda'}$ (resp. $\chi^{\lambda''}$) be the irreducible character of $S_{m'}$ (resp. $S_{m''}$) corresponding to the partition $\lambda'\in\mathscr{P}_{m'}$ (resp. $\lambda''\in\mathscr{P}_{m''}$). We denote by $\widetilde\chi^{\lambda'}$ the irreducible character of $W_{m'}=S_{m'}\ltimes(\mathbf{Z}/2\mathbf{Z})^{m'}$ obtained by extending $\chi^{\lambda'}$ by the trivial action of $(\mathbf{Z}/2\mathbf{Z})^{m'}$. We also denote by $\widetilde\chi^{\lambda''}$ the irreducible character of $W_{m''}=S_{m''}\ltimes(\mathbf{Z}/2\mathbf{Z})^{m''}$ by extending $\chi^{\lambda''}$ by defining the action of $(\mathbf{Z}/2\mathbf{Z})^{m''}$ so that each factor $\mathbf{Z}/2\mathbf{Z}$ acts non-trivially. Then $\operatorname{Ind}_{W_{m'}\times W_{m''}}^{W_n}\widetilde\chi^{\boldsymbol{\lambda}'}\otimes\widetilde\chi^{\boldsymbol{\lambda}''}$ gives an irreducible character $\chi^{\boldsymbol{\lambda}}$. It follows from the construction that $\chi^{\boldsymbol{\lambda}}|_{S_n}$ coincides with $\operatorname{Ind}_{S_{m'}\times S_{m''}}^{S_n}\chi^{\lambda'}\otimes\chi^{\lambda''}$.

For $\nu=(\nu_1,\dots,\nu_k)\in\mathscr{P}_n$, we denote by S_ν the Young subgroup $S_{\nu_1}\times\cdots\times S_{\nu_k}$. We show the following formula.

Proposition 3.15. *Let $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_{n,2}$ with $\boldsymbol{\mu}=(-,\mu'')$. Then we have*

$$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(1)=\langle\operatorname{Ind}_{S_{\mu''}}^{W_n}1,\chi^{\boldsymbol{\lambda}}\rangle_{W_n}. \tag{3.15.1}$$

Proof. Under the notation in 3.14, we compute the inner product.

$$
\begin{aligned}
\langle\operatorname{Ind}_{S_{\mu''}}^{W_n}1,\chi^{\boldsymbol{\lambda}}\rangle_{W_n} &= \langle\operatorname{Ind}_{S_{\mu''}}^{S_n}1,\chi^{\boldsymbol{\lambda}}|_{S_n}\rangle_{S_n} \\
&= \langle\operatorname{Ind}_{S_{\mu''}}^{S_n}1,\operatorname{Ind}_{S_{m'}\times S_{m''}}^{S_n}\chi^{\lambda'}\otimes\chi^{\lambda''}\rangle_{S_n}.
\end{aligned}
$$

Here we can write $\operatorname{Ind}_{S_{m'}\times S_{m''}}^{S_n}\chi^{\lambda'}\otimes\chi^{\lambda''}=\sum_{\nu\in\mathscr{P}_n}c^{\nu}_{\lambda'\lambda''}\chi^{\nu}$ by using the Littlewood-Richardson coefficients. Thus

$$\langle\operatorname{Ind}_{S_{\mu''}}^{W_n}1,\chi^{\boldsymbol{\lambda}}\rangle_{W_n}=\sum_{\nu\in\mathscr{P}_n}c^{\nu}_{\lambda'\lambda''}\langle\operatorname{Ind}_{S_{\mu''}}^{S_n}1,\chi^{\nu}\rangle_{S_n}.$$

But it is known that $\langle\operatorname{Ind}_{S_{\mu''}}^{S_n}1,\chi^{\nu}\rangle_{S_n}=K_{\nu,\mu''}(1)$ (see eg. [M, I, Remark after (7.8)]). Hence we have

$$\langle\operatorname{Ind}_{S_{\mu''}}^{W_n}1,\chi^{\boldsymbol{\lambda}}\rangle_{W_n}=\sum_{\nu\in\mathscr{P}_n}c^{\nu}_{\lambda'\lambda''}K_{\nu,\mu''}(1).$$

Then the proposition follows from (3.4.2), by substituting $t=1$. $\square$

Corollary 3.16. *Let $\boldsymbol{\mu} = (-;\mu'')$. Then for $z \in \mathscr{O}_{\boldsymbol{\mu}}$, we have*

$$\bigoplus_{i\ge 0} H^{2i}(\mathscr{B}_z, \bar{\mathbf{Q}}_l) \simeq \operatorname{Ind}_{S_{\mu''}}^{W_n} 1 \tag{3.16.1}$$

as W_n-modules.

Proof. Put $H^*(\mathscr{B}_z) = \bigoplus_{i\ge 0} H^{2i}(\mathscr{B}_z, \bar{\mathbf{Q}}_l)$. Then Proposition 2.14 shows that

$$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(1) = \langle H^*(\mathscr{B}_z), \chi^{\boldsymbol{\lambda}}\rangle_{W_n}$$

for any $\boldsymbol{\lambda} \in \mathscr{P}_{n,2}$. Thus, by comparing it with (3.15.1), we obtain the required formula. □

Remark 3.17. It would be interesting to compare (3.16.1) with a similar formula for the ordinary Springer representations of type C_n. We follow the setting in 2.11. For $x \in \mathfrak{g}^\theta_{\mathrm{nil}}$, we define

$$\mathscr{B}^\star_x = \{gB^\theta \in \mathscr{B} \mid g^{-1}x \in \operatorname{Lie} B^\theta\}.$$

$\mathscr{B}^\star_x$ is the original Springer fibre associated to $x \in \mathfrak{g}^\theta_{\mathrm{nil}}$, and the cohomology group $H^i(\mathscr{B}^\star_x, \bar{\mathbf{Q}}_l)$ has a natural action of W_n. It is known that $H^i(\mathscr{B}^\star_x, \bar{\mathbf{Q}}_l) = 0$ for odd i. Let $\mathfrak{l}^\theta$ be a Levi subalgebra of a parabolic subalgebra of $\mathfrak{g}^\theta$ of type $A_{\mu''_1-1} + A_{\mu''_2-1} + \cdots + A_{\mu''_k-1}$ for $\mu'' = (\mu''_1, \mu''_2, \dots, \mu''_k) \in \mathscr{P}_n$. Assume that x is a regualr nilpotent element in $\mathfrak{l}^\theta_{\mathrm{nil}}$. Then by a general formula due to [L2], we have

$$\bigoplus_{i\ge 0} H^{2i}(\mathscr{B}^\star_x, \bar{\mathbf{Q}}_l) \simeq \operatorname{Ind}_{S_{\mu''}}^{W_n} 1 \tag{3.17.1}$$

as W_n-modules. However, the graded W_n-module structures in (3.16.1) and (3.17.1) do not coincide in general. For example, assume that $n = 2$, and $\boldsymbol{\mu} = (-;2)$, i.e., $\mu'' = (2)$. In that case, $\operatorname{Ind}_{S_{\mu''}}^{W_2} 1 = \operatorname{Ind}_{S_2}^{W_2} 1 = \chi^{(-;2)} + \chi^{(1;1)} + \chi^{(2;-)}$. We have

$$\begin{aligned} &H^4(\mathscr{B}_z, \bar{\mathbf{Q}}_l) = \chi^{(-;2)}, \quad H^2(\mathscr{B}_z, \bar{\mathbf{Q}}_l) = \chi^{(1;1)}, \quad H^0(\mathscr{B}_z, \bar{\mathbf{Q}}_l) = \chi^{(2;-)}, \\ &H^2(\mathscr{B}^\star_x, \bar{\mathbf{Q}}_l) = \chi^{(-;2)} + \chi^{(1;1)}, \quad H^0(\mathscr{B}^\star_x, \bar{\mathbf{Q}}_l) = \chi^{(2;-)}. \end{aligned}$$

3.18. We shall give an interpretation of the formula (3.2.1) in terms of the Springer modules. Let $A_n = (\mathbf{Z}/2\mathbf{Z})^n$ be the abelian subgroup of W_n. We denote by $t_1, \dots, t_n$ the generators of A_n, where t_i is the generator of the i-th component $\mathbf{Z}/2\mathbf{Z}$. Let φ be a linear character of A_n. For each A_n-module X, we denote by X_φ the weight space of X corresponding to φ, namely $X_\varphi = \{v \in X \mid av = \varphi(a)v \text{ for } a \in A_n\}$. Let S_φ be the stabilizer of φ in S_n under the action of S_n on A_n. Then $S_\varphi \simeq S_m \times S_{n-m}$, where m is the number of i such that $\varphi(t_i) = 1$. If X is an W_n-module, X is an A_n-module by restriction. Then X_φ turns out to be an S_φ-module.

The W_n-module $H^i(\mathscr{B}_z, \bar{\mathbf{Q}}_l)$, which is called the (exotic) Springer modue, is isomorphic to each other for $z \in \mathscr{O}_{\boldsymbol{\mu}}$ ($\boldsymbol{\mu} \in \mathscr{P}_{n,2}$). In the discussion below, we denote it simply by $H^i(\mathscr{B}_{\boldsymbol{\mu}})$. Let $\mathscr{B}^0 = G_0/B_0$ be the flag variety of $G_0 = GL_n$, where B_0 is a Borel subgroup of G_0. Recall that for each nilpotent element $x \in \mathfrak{gl}_n$, the Springer fibre $\mathscr{B}^0_x$ is defined as

$$\mathscr{B}^0_x = \{gB_0 \in \mathscr{B}^0 \mid g^{-1}x \in \operatorname{Lie} B_0\},$$

and the cohomology group $H^i(\mathscr{B}^0_x, \bar{\mathbf{Q}}_l)$ has a natural strucutre of S_n-module, the Springer module. Since the S_n-module strucutre does not depend on $x \in \mathscr{O}_\nu$ ($\nu \in \mathscr{P}_n$), we denote it by $H^i(\mathscr{B}^0_\nu)$. Let $A_n^\wedge$ be the set of irreducible characters of A_n. Then we have the weight space decomposition

$$H^i(\mathscr{B}_{\boldsymbol{\mu}}) = \bigoplus_{\varphi \in A_n^\wedge} H^i(\mathscr{B}_{\boldsymbol{\mu}})_\varphi,$$

where each $H^i(\mathscr{B}_{\boldsymbol{\mu}})_\varphi$ has a structure of S_φ-module.

Recall the polynomial $g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t) \in \mathbf{Z}[t]$ for $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}_{n,2}$ given Proposition 3.2. We write it as

$$g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t) = \sum_{\ell \ge 0} g^{\boldsymbol{\mu}}_{\boldsymbol{\nu},\ell} t^\ell$$

with (possibly negative) integers $g^{\boldsymbol{\mu}}_{\boldsymbol{\nu},\ell}$. The following proposition gives a description of $H^i(\mathscr{B}_{\boldsymbol{\mu}})_\varphi$ in terms of the Springer modules of S_φ.

Proposition 3.19. *Assume that $\boldsymbol{\mu} \in \mathscr{P}_{n,2}$. Let $\varphi \in A_n^\wedge$ be such that $S_\varphi \simeq S_m \times S_{n-m}$. Then the following equality holds (in the Grothendieck group of the category of S_φ-modules)*

$$H^{2i}(\mathscr{B}_{\boldsymbol{\mu}})_\varphi = \sum_{\substack{\boldsymbol{\nu} = (\nu', \nu'') \in \mathscr{P}_{n,2} \\ |\nu'| = m}} \sum_{j,k,\ell} g^{\boldsymbol{\mu}}_{\boldsymbol{\nu},\ell} \big(H^{2j}(\mathscr{B}^0_{\nu'}) \otimes H^{2k}(\mathscr{B}^0_{\nu''})\big),$$

where the second sum is taken over all j,k,ℓ satisfying the condition

$$i = (n-m) + 2\ell + 2(j+k).$$

Proof. By Proposition 2.14, one can write (as an identity in the Grothendieck group of the category of S_φ-modules, extended by scalar to $\mathbf{Z}[t]$)

$$\sum_{i \ge 0} H^{2i}(\mathscr{B}_{\boldsymbol{\mu}})_\varphi t^i \simeq \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{n,2}} \widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)(V_{\boldsymbol{\lambda}})_\varphi \tag{3.19.1}$$

for each $\varphi \in A_n^\wedge$. Assume that $S_\varphi \simeq S_m \times S_{n-m}$. It follows from the explicit construction of $V_{\boldsymbol{\lambda}}$ in 3.14 that $(V_{\boldsymbol{\lambda}})_\varphi = 0$ unless $|\lambda'| = m, |\lambda''| = n-m$, and in that case, $(V_{\boldsymbol{\lambda}})_\varphi \simeq V_{\lambda'} \otimes V_{\lambda''}$ as $S_m \times S_{n-m}$-modules, where $V_{\lambda'}$ denotes the irreducible S_m-module corresponding to $\chi^{\lambda'}$, and similarly for $V_{\lambda''}$. By (3.2.1), the right hand side of (3.19.1) can be written as

$$
\begin{aligned}
t^{n-m} & \sum_{\substack{\lambda' \in \mathscr{P}_m \\ \lambda'' \in \mathscr{P}_{n-m}}} \sum_{\boldsymbol{\nu}=(\nu',\nu'') \in \mathscr{P}_{n,2}} g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t^2) \widetilde{K}_{\lambda',\nu'}(t^2) \widetilde{K}_{\lambda'',\nu''}(t^2) V_{\lambda'} \otimes V_{\lambda''} \\
&= t^{n-m} \sum_{\boldsymbol{\nu}} g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t^2) \biggl(\sum_{\lambda' \in \mathscr{P}_m} \widetilde{K}_{\lambda',\nu'}(t^2) V_{\lambda'} \biggr) \otimes \biggl(\sum_{\lambda'' \in \mathscr{P}_{n-m}} \widetilde{K}_{\lambda'',\nu''}(t^2) V_{\lambda''} \biggr) \\
&= t^{n-m} \sum_{\boldsymbol{\nu}} g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t^2) \biggl(\sum_{i \ge 0} H^{2i}(\mathscr{B}^0_{\nu'}) t^{2i} \biggr) \otimes \biggl(\sum_{i \ge 0} H^{2i}(\mathscr{B}^0_{\nu''}) t^{2i} \biggr),
\end{aligned}
$$

where the last equality follows from the formulas analogous to Proposition 2.14 in the case of GL_m and GL_{n-m}. By comparing the last expression with the left hand side of (3.19.1), we obtain the proposition. □

3.20. We consider $\varphi \in A_n^\wedge$ in the special case where $m = n$ or $m = 0$. Put $\varphi = \varphi_+$ (resp. $\varphi = \varphi_-$) if $m = n$ (resp. $m = 0$). In these cases, $S_\varphi \simeq S_n$, and we have a more precise description of the S_n-module $H^i(\mathscr{B}_{\boldsymbol{\mu}})_\varphi$ as follows. (Note that $H^i(\mathscr{B}_{\boldsymbol{\mu}})_{\varphi_+}$ coincides with the A_n-fixed point subspace of $H^i(\mathscr{B}_{\boldsymbol{\mu}})$. The formula (i) in the corollary should be compared with the result in [SSr], where the case of ordinary Springer representations of type C_n is discussed.)

Corollary 3.21. *Assume that* $\boldsymbol{\mu} = (\mu', \mu'') \in \mathscr{P}_{n,2}$.

(i) *There exists an isomorphism of* S_n*-modules*

$$
H^{2i}(\mathscr{B}_{\boldsymbol{\mu}})_{\varphi_+} \simeq \begin{cases} H^i(\mathscr{B}^0_{\mu'+\mu''}) & \text{if } i \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}
$$

(ii) $H^{2i}(\mathscr{B}_{\boldsymbol{\mu}})_{\varphi_-} = 0$ *unless* $\boldsymbol{\mu} = (-;\mu'')$. *Assume that* $\boldsymbol{\mu} = (-;\mu'')$. *There exists an isomorphism of* S_n*-modules*

$$
H^{2i}(\mathscr{B}_{\boldsymbol{\mu}})_{\varphi_-} \simeq \begin{cases} H^{i-n}(\mathscr{B}^0_{\mu''}) & \text{if } i \equiv n \pmod 2, \\ 0 & \text{otherwise}. \end{cases}
$$

Proof. Assume that $\varphi = \varphi_+$. In that case $(V_{\boldsymbol{\lambda}})_\varphi = 0$ unless $\boldsymbol{\lambda} = (\lambda';-)$, and in that case, $(V_{\boldsymbol{\lambda}})_\varphi \simeq V_{\lambda'}$ as S_n-modules. Moreover, if $\boldsymbol{\lambda} = (\lambda';-)$, we have $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = \widetilde{K}_{\lambda',\mu'+\mu''}(t^2)$ by Proposition 2.5 (ii). On the other hand, assume that $\varphi = \varphi_-$. Then we have $(V_{\boldsymbol{\lambda}})_\varphi = 0$ unless $\boldsymbol{\lambda} = (-;\lambda'')$, and in that case, $(V_{\boldsymbol{\lambda}})_\varphi \simeq V_{\lambda''}$ as S_n-modules. Moreover, by Proposition 2.5 (i), if $\boldsymbol{\lambda} = (-;\lambda'')$, $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = 0$ unless $\boldsymbol{\mu} = (-;\mu'')$, and in that case, $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = t^n \widetilde{K}_{\lambda'',\mu''}(t)$. Then the corollary follows from (3.19.1) by a similar discussion as in the proof of Proposition 3.19. □

3.22. Recall that the Hall-Littlewood function $P_{\boldsymbol{\lambda}}(x;t)$ is defined by two types of variables $x^{(1)}, x^{(2)}$. Here we consider a specialization of those variables. We denote by $P_{\boldsymbol{\lambda}}(x;t)|_{x=(y,y)}$ the function in $\varLambda[t]$ obtained by substituting $x^{(1)} = x^{(2)} = y$. We further consider the specialization of this function by putting $t = 1$, i.e.,

$P_{\boldsymbol{\lambda}}(x;1)|_{x=(y,y)}$. The following result shows that the behaviour of $P_{\boldsymbol{\lambda}}(x;t)$ at $t=1$ is quite different from that of ordinary Hall-Littlewood functions (cf. Remark 2.8).

Proposition 3.23. *Under the notation as above, we have*

$$P_{\boldsymbol{\mu}}(x;1)|_{x=(y,y)} = \begin{cases} m_{\mu''}(y) & \text{if } \boldsymbol{\mu} = (-;\mu''), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Assume that $\boldsymbol{\mu} = (-;\mu'')$. Since $P_{\boldsymbol{\mu}}(x;t) = P_{\mu''}(x^{(2)};t)$ for $\boldsymbol{\mu} = (-;\mu'')$ by Corollary 1.12, we have

$$P_{\boldsymbol{\mu}}(x;1)|_{x=(y,y)} = m_{\mu''}(y), \tag{3.23.1}$$

which shows the first equality.

By (1.2.1) and (1.2.2), for any $\lambda \in \mathscr{P}_n$, we have

$$s_\lambda(y) = \sum_{\mu\in\mathscr{P}_n} K_{\lambda,\mu}(1)m_\mu(y).$$

Also by substituting $t=1$ in the formula (3.3.3) and by using (1.2.1), we have, for any partitions μ,ν,

$$m_\mu(y)m_\nu(y) = \sum_{\lambda\in\mathscr{P}_n} f^\lambda_{\mu\nu}(1)m_\lambda(y).$$

Thus for $\boldsymbol{\lambda} = (\lambda';\lambda'') \in \mathscr{P}_{n,2}$, we have

$$\begin{aligned} s_{\boldsymbol{\lambda}}(x)|_{x=(y,y)} &= s_{\lambda'}(y)s_{\lambda''}(y) \\ &= \sum_{\nu'}\sum_{\nu''} K_{\lambda',\nu'}(1)K_{\lambda'',\nu''}(1)m_{\nu'}(y)m_{\nu''}(y) \\ &= \sum_{\mu''\in\mathscr{P}_n} m_{\mu''}(y)\sum_{\nu',\nu''} f^{\mu''}_{\nu'\nu''}(1)K_{\lambda',\nu'}(1)K_{\lambda'',\nu''}(1) \\ &= \sum_{\boldsymbol{\mu}=(-,\mu'')} K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(1)m_{\mu''}(y). \end{aligned} \tag{3.23.2}$$

The last equality follows from (3.4.1). On the other hand, by 1.6, we have

$$s_{\boldsymbol{\lambda}}(x)|_{x=(y,y)} = \sum_{\boldsymbol{\mu}\in\mathscr{P}_{n,2}} K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(1)P_{\boldsymbol{\mu}}(x;1)|_{x=(y,y)}. \tag{3.23.3}$$

Put $\mathscr{P}'_{n,2} = \{\boldsymbol{\mu} = (\mu',\mu'') \in \mathscr{P}_{n,2} \mid |\mu'| \neq 0\}$. Then (3.23.2) and (3.23.3), together with (3.23.1) imply that

(3.23.4) $$\sum_{\boldsymbol{\mu}\in\mathscr{P}'_{n,2}} K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(1)P_{\boldsymbol{\mu}}(x;1)|_{x=(y,y)} = 0$$

for any $\boldsymbol{\lambda}\in\mathscr{P}_{n,2}$. By Proposition 1.7, $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)=0$ unless $\boldsymbol{\mu}\le\boldsymbol{\lambda}$, and $K_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(t)=1$. Now the proposition follows from (3.23.4) by induction on the partial order $\le$ on $\mathscr{P}'_{n,2}$. The proposition is proved. □

4. Hall bimodule

4.1. Before going into detalis on the Hall bimodule, we show a preliminary result. In this section we fix a total order on $\mathscr{P}_{n,2}$ which is compatible with the partial order $\le$ on $\mathscr{P}_{n,2}$. For $\boldsymbol{\nu}=(\nu',\nu'')\in\mathscr{P}_{n,2}$, put $R_{\boldsymbol{\nu}}(x;t)=P_{\nu'}(x^{(1)},t^2)P_{\nu''}(x^{(2)},t^2)$. Then $\{R_{\boldsymbol{\nu}}\mid\boldsymbol{\nu}\in\mathscr{P}_{n,2}\}$ gives a basis of $\Xi^n[t]$. Hence there exist polynomials $h^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t)\in\mathbf{Z}[t]$ such that

(4.1.1) $$R_{\boldsymbol{\nu}}(x;t)=\sum_{\boldsymbol{\mu}\in\mathscr{P}_{n,2}} h^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t)P_{\boldsymbol{\mu}}(x;t).$$

The transition matrix between the bases $\{s_{\boldsymbol{\lambda}}\}$ and $\{R_{\boldsymbol{\nu}}\}$ is lower unitriangular, (with respect to the fixed total order) and a similar result holds also for the bases $\{s_{\boldsymbol{\lambda}}\}$ and $\{P_{\boldsymbol{\mu}}\}$. Hence the transition matrix $(h^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t))_{\boldsymbol{\mu},\boldsymbol{\nu}\in\mathscr{P}_{n,2}}$ between $\{R_{\boldsymbol{\nu}}\}$ and $\{P_{\boldsymbol{\mu}}\}$ is also lower unitriangular (we regard that the $\boldsymbol{\nu}\boldsymbol{\mu}$-entry is $h^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t)$). The following formula is an analogue of the formula (3.3.4) relating the polynomials $f^{\lambda}_{\mu\nu}(t)$ with the Hall polynomials $g^{\lambda}_{\mu\nu}(t)$.

Proposition 4.2. *Let $g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t)$ be the polynomials given in Propostion 3.2. Then*

(4.2.1) $$h^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t)=t^{a(\boldsymbol{\mu})-a(\boldsymbol{\nu})}g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t^{-2}).$$

In particular, the matrix $(g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t))_{\boldsymbol{\mu},\boldsymbol{\nu}}$ is lower unitriangular.

Proof. For any $\boldsymbol{\lambda}=(\lambda',\lambda'')\in\mathscr{P}_{n,2}$, we have

$$\begin{aligned}
s_{\boldsymbol{\lambda}}(x) &= s_{\lambda'}(x^{(1)})s_{\lambda''}(x^{(2)}) \\
&= \sum_{\nu'}K_{\lambda',\nu'}(t^2)P_{\nu'}(x^{(1)};t^2)\sum_{\nu''}K_{\lambda'',\nu''}(t^2)P_{\nu''}(x^{(2)};t^2) \\
&= \sum_{\nu',\nu''}K_{\lambda',\nu'}(t^2)K_{\lambda'',\nu''}(t^2)\sum_{\boldsymbol{\mu}\in\mathscr{P}_{n,2}}h^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t)P_{\boldsymbol{\mu}}(x;t) \\
&= \sum_{\boldsymbol{\mu}\in\mathscr{P}_{n,2}}\biggl(\sum_{\nu',\nu''}K_{\lambda',\nu'}(t^2)K_{\lambda'',\nu''}(t^2)h^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t)\biggr)P_{\boldsymbol{\mu}}(x;t).
\end{aligned}$$

Since

$$s_{\boldsymbol{\lambda}}(x) = \sum_{\boldsymbol{\mu}\in\mathscr{P}_{n,2}} K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)P_{\boldsymbol{\mu}}(x;t),$$

by comparing the coefficients of $P_{\boldsymbol{\mu}}(x;t)$, we have

$$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = \sum_{\nu',\nu''} h^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t)K_{\lambda',\nu'}(t^2)K_{\lambda'',\nu''}(t^2). \tag{4.2.2}$$

On the other hand, if we notice that $K_{\lambda'',\nu''}(t^2) \neq 0$ only when $|\lambda''| = |\nu''|$, the formual (3.3.1) can be rewritten as

$$K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) = \sum_{\nu',\nu''} t^{a(\boldsymbol{\mu})-a(\boldsymbol{\nu})} g^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(t^{-2})K_{\lambda',\nu'}(t^2)K_{\lambda'',\nu''}(t^2). \tag{4.2.3}$$

Since $(K_{\lambda',\nu'}(t^2)K_{\lambda'',\nu''}(t^2))_{\boldsymbol{\lambda},\boldsymbol{\nu}\in\mathscr{P}_{n,2}}$ is a unitriangular matrix with respect to the partial order on $\mathscr{P}_{n,2}$, the proposition is obtained by comparing (4.2.2) and (4.2.3). $\square$

4.3. We keep the assumption in 3.1, in particular, k is an algebraic closure of $\mathbf{F}_q$. Based on the idea of Finkelberg-Ginzburg-Travkin [FGT], we introduce the Hall bimodule. Let $\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathscr{P}_{n,2}, \alpha \in \mathscr{P}_n$, and take $(x,v) \in \mathscr{O}_{\boldsymbol{\lambda}}$. We define varieties

$$\begin{aligned}
\mathscr{G}^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}} = \{W \subset V \mid W &: x\text{-stable subspace}, \\
&x|_W : \text{type } \alpha, (x|_{V/W}, v \,(\mathrm{mod}\ W)) : \text{type } \boldsymbol{\mu}\}, \\
\mathscr{G}^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha} = \{W \subset V \mid W &: x\text{-stable subspace}, v \in W, \\
&(x|_W, v) : \text{type } \boldsymbol{\mu}\ , x|_{V/W} : \text{type } \alpha\}.
\end{aligned}$$

If $(x,v) \in \mathscr{O}_{\boldsymbol{\lambda}}(\mathbf{F}_q)$, those vatieties are defined over $\mathbf{F}_q$, and one can consider the subsets of $\mathbf{F}_q$-fixed points. Assuming that $(x,v) \in \mathscr{O}_{\boldsymbol{\lambda}}(\mathbf{F}_q)$, we define integers $G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(q)$ and $G^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(q)$ by

$$G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(q) = |\mathscr{G}^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(\mathbf{F}_q)|, \quad G^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(q) = |\mathscr{G}^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(\mathbf{F}_q)|. \tag{4.3.1}$$

Note that $G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(q), G^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(q)$ are independent of the choice of $(x,v) \in \mathscr{O}_{\boldsymbol{\lambda}}(\mathbf{F}_q)$. It is clear from the definition that $G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(q) = G^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(q) = 0$ unless $|\boldsymbol{\lambda}| = |\alpha| + |\boldsymbol{\mu}|$. In the case where $\boldsymbol{\lambda} = (-;\lambda''), \boldsymbol{\mu} = (-;\mu''), G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(q) = G^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(q)$ coincides with $g^{\lambda''}_{\mu'',\alpha}(q) = g^{\lambda''}_{\mu'',\alpha}|_{t=q}$, where $g^{\lambda''}_{\mu'',\alpha}$ is the original Hall polynomial given in 3.3.

Put $\mathscr{P} = \coprod_{n\geq 0}\mathscr{P}_n$ and $\mathscr{P}^{(2)} = \coprod_{n\geq 0}\mathscr{P}_{n,2}$. Recall that the definition of the Hall algebea $\mathscr{H}$; $\mathscr{H}$ is the free $\mathbf{Z}[t]$-module with basis $\{\mathfrak{u}_\alpha \mid \alpha \in \mathscr{P}\}$, and the

multiplication is defined by

$$\mathfrak{u}_\beta \mathfrak{u}_\gamma = \sum_{\alpha \in \mathscr{P}_n} g^\alpha_{\beta,\gamma}(t)\mathfrak{u}_\alpha,$$

where $n = |\beta| + |\gamma|$. $\mathscr{H}$ is a commutative, associative algebra over $\mathbf{Z}[t]$. We define the $\mathbf{Z}$-algebra $\mathscr{H}_q$ by $\mathscr{H}_q = \mathbf{Z} \otimes_{\mathbf{Z}[t]} \mathscr{H}$, under the specialization $\mathbf{Z}[t] \to \mathbf{Z}$, $t \mapsto q$.

We define a Hall bimodule $\mathscr{M}_q$ as follows; $\mathscr{M}_q$ is a free $\mathbf{Z}$-module with basis $\{\mathfrak{u}_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}^{(2)}\}$. We define actions (the left action and the rignt acton) of $\mathscr{H}_q$ on $\mathscr{M}_q$ by

$$\mathfrak{u}_\alpha \mathfrak{u}_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{n,2}} G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(q)\mathfrak{u}_{\boldsymbol{\lambda}}, \tag{4.3.2}$$

$$\mathfrak{u}_{\boldsymbol{\mu}} \mathfrak{u}_\alpha = \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{n,2}} G^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(q)\mathfrak{u}_{\boldsymbol{\lambda}}, \tag{4.3.3}$$

where $n = |\alpha| + |\boldsymbol{\mu}|$. Then $\mathscr{M}_q$ turns out to be a $\mathscr{H}_q$-bimodule, which is verified as follows; for partitions β, γ, and double partitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$, we define a variety

$$\begin{aligned}\mathscr{G}^{\boldsymbol{\lambda}}_{\beta,\gamma;\boldsymbol{\mu}} = \{(W_1 \subset W_2) \mid W_1, W_2 :\ & x\text{-stable subspaces of } V, \\ & x_{W_1}\text{: type } \beta,\ x_{W_2/W_1} : \text{type } \gamma,\ (x_{V/W_2}, v \pmod{W_2}) : \text{type } \boldsymbol{\mu}\ \}.\end{aligned}$$

We compute the number $|\mathscr{G}^{\boldsymbol{\lambda}}_{\beta,\gamma;\boldsymbol{\mu}}(\mathbf{F}_q)|$ in two different ways. Put $n = |\beta| + |\gamma|$. Assume that x_{W_2} has type α. Then the cardinality of such W_2 is given by $G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(q)$. For each W_2, the cardinality of W_1 is given by $g^\alpha_{\beta,\gamma}(q)$. It follows that

$$|\mathscr{G}^{\boldsymbol{\lambda}}_{\beta,\gamma;\boldsymbol{\mu}}(\mathbf{F}_q)| = \sum_{\alpha \in \mathscr{P}_n} g^\alpha_{\beta,\gamma}(q) G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(q). \tag{4.3.4}$$

On the other hand, the cardinality of W_1 satisfying the condition that $x|_{W_1}$ has type β, $(x|_{V/W_1}, v \pmod{W_1})$ has type $\boldsymbol{\nu}$ is $G^{\boldsymbol{\lambda}}_{\beta,\boldsymbol{\nu}}(q)$. For each W_1, the cardinality of W_2 such that $W_1 \subset W_2 \subset V$ and that $x|_{W_2/W_1}$ has type γ, $(x|_{V/W_2}, v \pmod{W_2})$ has type $\boldsymbol{\mu}$ is given by $G^{\boldsymbol{\nu}}_{\gamma,\boldsymbol{\mu}}(q)$. It follows that

$$|\mathscr{G}^{\boldsymbol{\lambda}}_{\beta,\gamma;\boldsymbol{\mu}}(\mathbf{F}_q)| = \sum_{\boldsymbol{\nu} \in \mathscr{P}_{m,2}} G^{\boldsymbol{\lambda}}_{\beta,\boldsymbol{\nu}}(q) G^{\boldsymbol{\nu}}_{\gamma,\boldsymbol{\mu}}(q), \tag{4.3.5}$$

where $m = |\boldsymbol{\lambda}| - |\beta|$. Now the equality (4.3.4) = (4.3.5) implies that $\mathfrak{u}_\beta(\mathfrak{u}_\gamma \mathfrak{u}_{\boldsymbol{\mu}}) = (\mathfrak{u}_\beta \mathfrak{u}_\gamma)\mathfrak{u}_{\boldsymbol{\mu}}$. In a similar way, one can show that $(\mathfrak{u}_{\boldsymbol{\mu}} \mathfrak{u}_\gamma)\mathfrak{u}_\beta = \mathfrak{u}_{\boldsymbol{\mu}}(\mathfrak{u}_\gamma \mathfrak{u}_\beta)$. Next we consider a variety

$$\begin{aligned}\mathscr{G}^{\boldsymbol{\lambda}}_{\alpha;\boldsymbol{\mu};\beta} = \{(W_1 \subset W_2) \mid\ & W_1, W_2\ x\text{-stable subspaces of } V, v \in W_2 \\ & x|_{W_1} : \text{type } \alpha, , (x_{W_2/W_1}, v \pmod{W_1}) : \text{type } \boldsymbol{\mu},\ x_{V/W_2} : \text{type } \beta\}.\end{aligned}$$

We compute the number $|\mathscr{G}^{\boldsymbol{\lambda}}_{\alpha;\boldsymbol{\mu};\beta}(\mathbf{F}_q)|$ in two different ways. Take $W_2 \in \mathscr{G}^{\boldsymbol{\lambda}}_{\boldsymbol{\nu},\beta}(\mathbf{F}_q)$ for some $\boldsymbol{\nu} \in \mathscr{P}_{n,2}$ with $n = |\boldsymbol{\lambda}| - |\beta|$. The cardinality of such W_2 is $G^{\boldsymbol{\lambda}}_{\boldsymbol{\nu},\beta}(q)$. For each W_2, the cardinality of W_1 such that $(W_1 \subset W_2) \in \mathscr{G}^{\boldsymbol{\lambda}}_{\alpha;\boldsymbol{\mu};\beta}(\mathbf{F}_q)$ is given by $G^{\boldsymbol{\nu}}_{\alpha,\boldsymbol{\mu}}(q)$. Thus

$$|\mathscr{G}^{\boldsymbol{\lambda}}_{\alpha;\boldsymbol{\mu};\beta}(\mathbf{F}_q)| = \sum_{\boldsymbol{\nu}\in\mathscr{P}_{n,2}} G^{\boldsymbol{\lambda}}_{\boldsymbol{\nu},\beta}(q)G^{\boldsymbol{\nu}}_{\alpha,\boldsymbol{\mu}}(q).$$

On the other hand, first we take $W_1 \in \mathscr{G}^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\nu}}(\mathbf{F}_q)$, and then take W_2 such that $(W_1 \subset W_2)$ is contained in $\mathfrak{S}^{\boldsymbol{\lambda}}_{\alpha;\boldsymbol{\mu};\beta}(\mathbf{F}_q)$. This implies that

$$|\mathscr{G}^{\boldsymbol{\lambda}}_{\alpha;\boldsymbol{\mu};\beta}(\mathbf{F}_q)| = \sum_{\boldsymbol{\nu}\in\mathscr{P}_{n',2}} G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\nu}}(q)G^{\boldsymbol{\nu}}_{\boldsymbol{\mu},\beta}(q),$$

where $n' = |\boldsymbol{\lambda}| - |\alpha|$. Comparing these two equalities, we have $\mathfrak{u}_\alpha(\mathfrak{u}_{\boldsymbol{\mu}}\mathfrak{u}_\beta) = (\mathfrak{u}_\alpha\mathfrak{u}_{\boldsymbol{\mu}})\mathfrak{u}_\beta$. Thus $\mathscr{M}_q$ has a structure of $\mathscr{H}_q$-bimodule.

For an integer $n \ge 0$, let $\mathscr{M}^n_q$ be the $\mathbf{Z}$-submodule of $\mathscr{M}_q$ spanned by $\mathfrak{u}_{\boldsymbol{\lambda}}$ with $\boldsymbol{\lambda} \in \mathscr{P}_{n,2}$. Then we have $\mathscr{M}_q = \bigoplus_{n\ge0}\mathscr{M}^n_q$. Similarly, we have a decompositon $\mathscr{H}_q = \bigoplus_{n\ge0}\mathscr{H}^n_q$. The above discussion shows that $\mathscr{M}_q$ has a structure of graded $\mathscr{H}_q$-bimodule, i.e., $\mathscr{H}^m_q\mathscr{M}^n_q \subset \mathscr{M}^{n+m}_q$, and $\mathscr{M}^n_q\mathscr{H}^m_q \subset \mathscr{M}^{n+m}_q$.

4.4. For $\boldsymbol{\lambda} = (-;-)$, put $\mathfrak{u}_0 = \mathfrak{u}_{\boldsymbol{\lambda}}$. It is easy to see that $\mathfrak{u}_0\mathfrak{u}_\beta = \mathfrak{u}_{(-;\beta)}$ for $\beta \in \mathscr{P}$ (but $\mathfrak{u}_\beta\mathfrak{u}_0 \ne \mathfrak{u}_{(\beta;-)}$). Take $\alpha, \beta \in \mathscr{P}$. One can check that $G^{\boldsymbol{\lambda}}_{\alpha;(-;\beta)}(q) = g^{\boldsymbol{\lambda}}_{(\alpha;\beta)}(q)$ for $\boldsymbol{\lambda} \in \mathscr{P}^{(2)}$. It follows, for $\alpha, \beta \in \mathscr{P}$, that

$$\mathfrak{u}_\alpha\mathfrak{u}_0\mathfrak{u}_\beta = \sum_{\boldsymbol{\lambda}\in\mathscr{P}_{n,2}} g^{\boldsymbol{\lambda}}_{(\alpha;\beta)}(q)\mathfrak{u}_{\boldsymbol{\lambda}}, \tag{4.4.1}$$

where $n = |\alpha| + |\beta|$. For each $\boldsymbol{\mu} = (\mu';\mu'') \in \mathscr{P}_{n,2}$, put $\mathfrak{v}_{\boldsymbol{\mu}} = \mathfrak{u}_{\mu'}\mathfrak{u}_0\mathfrak{u}_{\mu''}$. We have a lemma.

Lemma 4.5. *$\{\mathfrak{v}_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_{n,2}\}$ gives a basis of $\mathscr{M}^n_q$. Hence $\{\mathfrak{v}_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}^{(2)}\}$ gives a basis of $\mathscr{M}_q$. For $\boldsymbol{\mu} \in \mathscr{P}_{n,2}$, we have*

$$\mathfrak{v}_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\lambda}\in\mathscr{P}_{n,2}} g^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(q)\mathfrak{u}_{\boldsymbol{\lambda}}. \tag{4.5.1}$$

In particular, $\mathscr{M}_q$ is a free $\mathscr{H}_q$-bimodule of rank 1 (with a basis $\mathfrak{v}_{(-;-)} = \mathfrak{u}_0$).

Proof. (4.5.1) follows from (4.4.1). $\mathscr{M}^n_q$ is a free $\mathbf{Z}$-module with rank $|\mathscr{P}_{n,2}|$. By Proposition 4.2, $(g^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(q))_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_{n,2}}$ is a unitriangular matrix with respect to a certain total order on $\mathscr{P}_{n,2}$. Thus $\{\mathfrak{v}_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_{n,2}\}$ gives rise to a basis of $\mathscr{M}^n_q$. □

4.6. Recall that $\Xi = \varLambda(x^{(1)}) \otimes \varLambda(x^{(2)})$, and $\Xi[t] = \varLambda(x^{(1)})[t] \otimes_{\mathbf{Z}[t]} \varLambda(x^{(2)})[t]$. Thus $\Xi[t]$ is regarded as a free $\varLambda[t]$-bimodule of rank 1 ($\varLambda = \varLambda(y)$ acts on $\varLambda(x^{(1)})$ by replacing y by $x^{(1)}$, and so on for $\varLambda(x^{(2)})$). It is known by [M, III, (3.4)] that the map $\mathfrak{u}_\alpha \mapsto t^{-n(\alpha)}P_\alpha(y;t^{-1})$ gives an isomorphism of rings $\mathscr{H} \otimes \mathbf{Z}[t,t^{-1}] \xrightarrow{\sim} \varLambda \otimes \mathbf{Z}[t,t^{-1}]$.

This induces an isomorpism $\mathscr{H}_q \otimes \mathbf{Q} \xrightarrow{\sim} \Lambda_{\mathbf{Q}}$. We define a map $\Psi : \mathscr{M}_{q^2} \otimes \mathbf{Q} \to \Xi_{\mathbf{Q}}$ by

$$\mathfrak{v}_{\boldsymbol{\mu}} \mapsto q^{-a(\boldsymbol{\mu})} P_{\mu'}(x^{(1)}, q^{-2}) P_{\mu''}(x^{(2)}, q^{-2}) = q^{-a(\boldsymbol{\mu})} R_{\boldsymbol{\mu}}(x; q^{-1}) \tag{4.6.1}$$

for $\boldsymbol{\mu} = (\mu', \mu'') \in \mathscr{P}^{(2)}$. Then it is clear that Ψ gives an isomorphism $\mathscr{M}_{q^2} \otimes \mathbf{Q} \xrightarrow{\sim} \Xi_{\mathbf{Q}}$ of bimodules (under the isomorphism $\mathscr{H}_{q^2} \otimes \mathbf{Q} \xrightarrow{\sim} \Lambda_{\mathbf{Q}}$).

By making use of (4.2.1), the formula (4.5.1) can be rewritten as

$$q^{a(\boldsymbol{\mu})}\mathfrak{v}_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{n,2}} h^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(q^{-1}) q^{a(\boldsymbol{\lambda})} \mathfrak{u}_{\boldsymbol{\lambda}},$$

where $\mathfrak{v}_{\boldsymbol{\mu}}, \mathfrak{u}_{\boldsymbol{\lambda}} \in \mathscr{M}_{q^2}$. Since $(h^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(q))_{\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathscr{P}_{n,2}}$ is the transition matrix between the bases $\{R_{\boldsymbol{\mu}}(x;q)\}$ and $\{P_{\boldsymbol{\lambda}}(x;q)\}$ of $\Xi^n_{\mathbf{Q}}$, we see that

$$\Psi(\mathfrak{u}_{\boldsymbol{\lambda}}) = q^{-a(\boldsymbol{\lambda})} P_{\boldsymbol{\lambda}}(x; q^{-1}). \tag{4.6.2}$$

For given $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^{(2)}$, $\alpha \in \mathscr{P}$, we define polynomials $H^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(t), H^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(t) \in \mathbf{Z}[t]$ by

$$\begin{aligned}
P_\alpha(x^{(1)}; t^2) P_{\boldsymbol{\mu}}(x;t) &= \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{n,2}} H^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(t) P_{\boldsymbol{\lambda}}(x;t), \\
P_{\boldsymbol{\mu}}(x;t) P_\alpha(x^{(2)}; t^2) &= \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{n,2}} H^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(t) P_{\boldsymbol{\lambda}}(x;t).
\end{aligned}$$

where $n = |\alpha| + |\boldsymbol{\mu}|$. Considering Ψ^{-1}, and by comparing (4.3.2) and (4.3.3), we have the following formula; for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^{(2)}, \alpha \in \mathscr{P}$,

$$G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(q^2) = q^{a(\boldsymbol{\lambda}) - a(\boldsymbol{\mu}) - 2n(\alpha)} H^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(q^{-1}), \tag{4.6.3}$$

$$G^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(q^2) = q^{a(\boldsymbol{\lambda}) - a(\boldsymbol{\mu}) - 2n(\alpha)} H^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(q^{-1}). \tag{4.6.4}$$

The following result can be compared with that of the mirabolic Hall bimodule in [FGT, §4].

Theorem 4.7. *Assume that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^{(2)}, \alpha \in \mathscr{P}$.*

(i) *There exist polynomials $G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}, G^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha} \in \mathbf{Z}[t]$ such that $G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(q) = G^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}|_{t=q}$, $G^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(q) = G^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}|_{t=q}$. Thus one can define a $\mathscr{H}_t$-bimodule structure for the free $\mathbf{Z}[t]$-module $\mathscr{M}_t = \bigoplus_{\boldsymbol{\lambda} \in \mathscr{P}^{(2)}} \mathbf{Z}[t]\mathfrak{u}_{\boldsymbol{\lambda}}$ by extending (4.3.2) and (4.3.3), where $\mathscr{H}_t$ denotes the Hall algebra $\mathscr{H}$ over $\mathbf{Z}[t]$.*

(ii) *$\mathscr{M}_t$ is a free $\mathscr{H}_t$-bimodule of rank 1, with the basis $\mathfrak{u}_0$. More precisely, let $\{\mathfrak{u}_\alpha \mid \alpha \in \mathscr{P}\}$ be the basis of $\mathscr{H}_t$. Then $\{\mathfrak{u}_{\mu'}\mathfrak{u}_0\mathfrak{u}_{\mu''} \mid (\mu';\mu'') \in \mathscr{P}^{(2)}\}$ gives a*

basis of $\mathscr{M}_t$. *For any* $\boldsymbol{\mu} = (\mu'; \mu'') \in \mathscr{P}_{n,2}$, *we have*

$$\mathfrak{u}_{\mu'}\mathfrak{u}_0\mathfrak{u}_{\mu''} = \sum_{\boldsymbol{\lambda}\in\mathscr{P}_{n,2}} g^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(t)\mathfrak{u}_{\boldsymbol{\lambda}}.$$

(iii) *The map* $\Psi : \mathfrak{u}_{\boldsymbol{\lambda}} \mapsto t^{-a(\boldsymbol{\lambda})}P_{\boldsymbol{\lambda}}(x; t^{-1})$ *gives an isomorphism*

$$\mathscr{M}_{t^2} \otimes_{\mathbf{Z}[t^2]} \mathbf{Z}[t, t^{-1}] \xrightarrow{\sim} \Xi \otimes \mathbf{Z}[t, t^{-1}]$$

as bimodules (under the isomorphism $\mathscr{H}_{t^2} \otimes_{\mathbf{Z}[t^2]} \mathbf{Z}[t, t^{-1}] \simeq \Lambda \otimes \mathbf{Z}[t, t^{-1}]$*).*

Proof. In view of (4.6.3) and (4.6.4), what we need to show is, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^{(2)}, \alpha \in \mathscr{P}$,

$$t^{a(\boldsymbol{\lambda})-a(\boldsymbol{\mu})-2n(\alpha)}H^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(t^{-1}) \in \mathbf{Z}[t^2], \tag{4.7.1}$$

$$t^{a(\boldsymbol{\lambda})-a(\boldsymbol{\mu})-2n(\alpha)}H^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\alpha}(t^{-1}) \in \mathbf{Z}[t^2]. \tag{4.7.2}$$

All other assertions follow from the discussion in 4.6. By (4.2.1), we see that $t^{a(\boldsymbol{\lambda})-a(\boldsymbol{\mu})}h^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(t^{-1}) \in \mathbf{Z}[t^2]$. The matrix $H(t^{-1}) = (h^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(t^{-1}))$ is unitriangular. Let $D(t)$ be the diagonal matrix such that the $\boldsymbol{\lambda}\boldsymbol{\lambda}$-entry is $t^{a(\boldsymbol{\lambda})}$. Then the matrix $(t^{a(\boldsymbol{\lambda})-a(\boldsymbol{\mu})}h^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}}(t^{-1}))$ coincides with $D(t)^{-1}H(t^{-1})D(t)$. This matrix is also unitriangular. It follows that each entry of its inverse matrix is contained in $\mathbf{Z}[t^2]$. Let $H(t^{-1})^{-1} = (h'_{\boldsymbol{\mu},\boldsymbol{\nu}}(t^{-1}))$ be the inverse matrix of $H(t^{-1})$. Then $t^{a(\boldsymbol{\nu})-a(\boldsymbol{\mu})}h'_{\boldsymbol{\mu},\boldsymbol{\nu}}(t^{-1}) \in \mathbf{Z}[t^2]$. Note that $H(t)$ is the transition matrix between the bases $\{R_{\boldsymbol{\mu}}\}$ and $\{P_{\boldsymbol{\lambda}}\}$. Hence $H(t)^{-1}$ is the transition matrix between the bases $\{P_{\boldsymbol{\mu}}\}$ and $\{R_{\boldsymbol{\nu}}\}$. One can write

$$P_{\boldsymbol{\mu}}(x; t) = \sum_{\boldsymbol{\nu}=(\nu',\nu'')\in\mathscr{P}^{(2)}} h'_{\boldsymbol{\mu},\boldsymbol{\nu}}(t)P_{\nu'}(x^{(1)}; t^2)P_{\nu''}(x^{(2)}; t^2).$$

Since

$$P_\alpha(x^{(1)}; t^2)P_{\nu'}(x^{(1)}; t^2) = \sum_{\xi\in\mathscr{P}} f^{\xi}_{\alpha,\nu'}(t^2)P_\xi(x^{(1)}; t^2),$$

we have

$$\begin{aligned} P_\alpha(x^{(1)}; t^2)P_{\boldsymbol{\mu}}(x; t) &= \sum_{\boldsymbol{\nu}\in\mathscr{P}^{(2)}} h'_{\boldsymbol{\mu},\boldsymbol{\nu}}(t)\sum_{\xi\in\mathscr{P}} f^{\xi}_{\alpha,\nu'}(t^2)P_\xi(x^{(1)}; t^2)P_{\nu''}(x^{(2)}; t^2) \\ &= \sum_{\boldsymbol{\nu},\xi} h'_{\boldsymbol{\mu},\boldsymbol{\nu}}(t)f^{\xi}_{\alpha,\nu'}(t^2)\sum_{\boldsymbol{\lambda}\in\mathscr{P}^{(2)}} h^{\boldsymbol{\lambda}}_{(\xi;\nu'')}(t)P_{\boldsymbol{\lambda}}(x; t). \end{aligned}$$

It follows that

$$H^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(t) = \sum_{\boldsymbol{\nu},\xi} h'_{\boldsymbol{\mu},\boldsymbol{\nu}}(t)f^{\xi}_{\alpha,\nu'}(t^2)h^{\boldsymbol{\lambda}}_{(\xi;\nu'')}(t). \tag{4.7.3}$$

Here $h'_{\boldsymbol{\mu},\boldsymbol{\nu}}(t^{-1}) \in t^{a(\boldsymbol{\mu})-a(\boldsymbol{\nu})}\mathbf{Z}[t^2]$ and $h^{\boldsymbol{\lambda}}_{(\xi,\nu'')}(t^{-1}) \in t^{a((\xi;\nu''))-a(\boldsymbol{\lambda})}\mathbf{Z}[t^2]$. Moreover, by (3.3.4), $f^{\xi}_{\alpha,\nu'}(t^{-2}) \in t^{2n(\alpha)+2n(\nu')-2n(\xi)}\mathbf{Z}[t^2]$. Since $a((\xi;\nu'')) = 2n(\xi) + 2n(\nu'') + |\nu''|$ and $a(\boldsymbol{\nu}) = 2n(\nu') + 2n(\nu'') + |\nu''|$, we see that $H^{\boldsymbol{\lambda}}_{\alpha,\boldsymbol{\mu}}(t^{-1}) \in t^{a(\boldsymbol{\mu})+2n(\alpha)-a(\boldsymbol{\lambda})}\mathbf{Z}[t^2]$. This proves (4.7.1). The proof of (4.7.2) is similar. $\square$

Appendix Tables of double Kostka polynomials

Let $K(t) = (K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t))_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathscr{P}_{n,2}}$ be the matrix of double Kostka polynomials. We give the table of matrices $K(t)$ for $2 \le n \le 5$. In the table below, we use the following notation; we denote the double partition $(\lambda;\mu)$ with $\lambda = (\lambda_1^{m_1}, \dots, \lambda_k^{m_k})$, $\mu = (\mu_1^{n_1}, \dots, \mu_{k'}^{n_{k'}})$ by $\lambda_1^{m_1}\cdots\lambda_k^{m_k}.\mu_1^{n_1}\cdots\mu_{k'}^{n_{k'}}$. For example,

$$(21^2;3^2) \leftrightarrow 21^2.3^2 \quad (32;-) \leftrightarrow 32. \quad (-;21^2) \leftrightarrow .21^2$$

and so on.

Table 1. $K(t)$ for $n = 2$

	2.	1.1	.2	1^2.	$.1^2$
2.	1	t	t^2	t^2	t^4
1.1		1	t	t	t^3+t
.2			1		t^2
1^2.				1	t^2
$.1^2$					1

Table 2. $K(t)$ for $n = 3$

	3.	2.1	1.2	21.	$1^2.1$	.3	1.1^2	.21	1^3.	$.1^3$
3.	1	t	t^2	t^2	t^3	t^3	t^4	t^5	t^6	t^9
2.1		1	t	t	t^2	t^2	t^3+t	t^4+t^2	t^5+t^3	$t^8+t^6+t^4$
1.2			1		t	t	t^2	t^3+t	t^4	$t^7+t^5+t^3$
21.				1	t		t^2	t^3	t^4+t^2	t^7+t^5
$1^2.1$					1		t	t^2	t^3+t	$t^6+t^4+t^2$
.3						1		t^2		t^6
1.1^2							1	t	t^2	t^5+t^3+t
.21								1		t^4+t^2
1^3.									1	t^3
$.1^3$										1

TABLE 3. $K(t)$ for $n = 4$

	4.	3.1	31.	2.2	21.1	1.3	2.1^2	$1^2.2$	$2^2.$	1.21	.4	$21^2.$	$1^2.1^2$	.31
4.	1	t	t^2	t^2	t^3	t^3	t^4	t^4	t^4	t^5	t^4	t^6	t^6	t^6
3.1		1	t	t	t^2	t^2	t^3+t	t^3	t^3	t^4+t^2	t^3	t^5+t^3	t^5+t^3	t^5+t^3
31.			1		t		t^2	t^2	t^2	t^3		t^4+t^2	t^4	t^4
2.2				1	t	t	t^2	t^2	t^2	t^3+t	t^2	t^4	t^4+t^2	t^4+t^2
21.1					1		t	t	t	t^2		t^3+t	t^3+t	t^3
1.3						1		t		t^2	t		t^3	t^3+t
2.1^2							1			t		t^2	t^2	t^2
$1^2.2$								1		t			t^2	t^2
$2^2.$									1			t^2	t^2	
1.21										1			t	t
.4											1			t^2
$21^2.$												1		
$1^2.1^2$													1	
.31														1
$1^3.1$														
$.2^2$														
1.1^3														
$.21^2$														
$1^4.$														
$.1^4$														

	$1^3.1$	$.2^2$	1.1^3	$.21^2$	$1^4.$	$.1^4$
4.	t^7	t^8	t^9	t^{10}	t^{12}	t^{16}
3.1	t^6+t^4	t^7+t^5	$t^8+t^6+t^4$	$t^9+t^7+t^5$	$t^{11}+t^9+t^7$	$t^{15}+t^{13}+t^{11}+t^9$
31.	t^5+t^3	t^6	t^7+t^5	t^8+t^6	$t^{10}+t^8+t^6$	$t^{14}+t^{12}+t^{10}$
2.2	t^5+t^3	$t^6+t^4+t^2$	$t^7+t^5+t^3$	$t^8+t^6+2t^4$	$t^{10}+t^8+t^6$	$t^{14}+t^{12}+2t^{10}+t^8+t^6$
21.1	t^4+2t^2	t^5+t^3	$t^6+2t^4+t^2$	$t^7+2t^5+t^3$	$t^9+2t^7+2t^5+t^3$	$t^{13}+2t^{11}+2t^9+2t^7+t^5$
1.3	t^4	t^5+t^3	t^6	$t^7+t^5+t^3$	t^9	$t^{13}+t^{11}+t^9+t^7$
2.1^2	t^3	t^4	t^5+t^3+t	$t^6+t^4+t^2$	$t^8+t^6+t^4$	$t^{12}+t^{10}+2t^8+t^6+t^4$
$1^2.2$	t^3+t	t^4	t^5+t^3	$t^6+t^4+t^2$	$t^8+t^6+t^4$	$t^{12}+t^{10}+2t^8+t^6+t^4$
$2^2.$	t^3	t^4	t^5	t^6	t^8+t^4	$t^{12}+t^8$
1.21	t^2	t^3+t	t^4+t^2	t^5+2t^3+t	t^7+t^5	$t^{11}+2t^9+2t^7+2t^5+t^3$
.4		t^4		t^6		t^{12}
$21^2.$	t		t^3	t^4	$t^6+t^4+t^2$	$t^{10}+t^8+t^6$
$1^2.1^2$	t	t^2	t^3+t	t^4+t^2	$t^6+t^4+t^2$	$t^{10}+t^8+2t^6+t^4+t^2$
.31		t^2		t^4+t^2		$t^{10}+t^8+t^6$
$1^3.1$	1		t^2	t^3	t^5+t^3+t	$t^9+t^7+t^5+t^3$
$.2^2$		1		t^2		t^8+t^4
1.1^3			1	t	t^3	$t^7+t^5+t^3+t$
$.21^2$				1		$t^6+t^4+t^2$
$1^4.$					1	t^4
$.1^4$						1

TABLE 4. $K(t)$ for $n = 5$

	5.	4.1	3.2	41.	2.3	31.1	1.4	21.2	3.1^2	32.	$1^2.3$	2.21	$2^2.1$	1.31	21.1^2	$31^2.$	$1^2.21$	$21^2.1$	.5
5.	1	t	t^2	t^2	t^3	t^3	t^4	t^4	t^4	t^4	t^5	t^5	t^5	t^6	t^6	t^6	t^7	t^7	t^5
4.1		1	t	t	t^2	t^2	t^3	t^3	t^3+t	t^3	t^4	t^4+t^2	t^4	t^5+t^3	t^5+t^3	t^5+t^3	t^6+t^4	t^6+t^4	t^4
3.2			1		t	t	t^2	t^2	t^2	t^2	t^3	t^3+t	t^3	t^4+t^2	t^4+t^2	t^4	t^5+t^3	t^5+t^3	t^3
41.				1		t		t^2	t^2	t^2	t^3	t^3	t^3	t^4	t^4	t^4+t^2	t^5	t^5+t^3	
2.3					1		t	t			t^2	t^2	t^2	t^3+t	t^3		t^4+t^2	t^4	t^2
31.1						1		t	t	t	t^2	t^2	t^2	t^3	t^3+t	t^3+t	t^4+t^2	t^4+2t^2	
1.4							1				t			t^2			t^3		t
21.2								1			t	t	t	t^2	t^2		t^3+t	t^3+t	
3.1^2									1			t		t^2	t^2	t^2	t^3	t^3	
32.										1			t		t^2	t^2	t^3	t^3	
$1^2.3$											1			t			t^2		
2.21												1		t	t		t^2	t^2	
$2^2.1$													1		t		t^2	t^2	
1.31														1			t		
21.1^2															1		t	t	
$31^2.$																1		t	
$1^2.21$																	1		
$21^2.1$																		1	
.5																			1
1.2^2																			
2.1^3																			
$1^3.2$																			
$2^21.$																			
.41																			
1.21^2																			
$1^3.1^2$																			
.32																			
$21^3.$																			
$1^2.1^3$																			
$.31^2$																			
$1^4.1$																			
$.2^21$																			
1.1^4																			
$.21^3$																			
$1^5.$																			
$.1^5$																			

	1.2^2	2.1^3	$1^3.2$	$2^21.$	$.41$	1.21^2	$1^3.1^2$	$.32$	$21^3.$	$1^2.1^3$
$5.$	t^8	t^9	t^8	t^8	t^7	t^{10}	t^{10}	t^9	t^{12}	t^{11}
4.1	t^7+t^5	$t^8+t^6+t^4$	t^7+t^5	t^7+t^5	t^6+t^4	$t^9+t^7+t^5$	t^9+t^7	t^8+t^6	$t^{11}+t^9+t^7$	$t^{10}+t^8+t^6$
3.2	$t^6+t^4+t^2$	$t^7+t^5+t^3$	t^6+t^4	t^6+t^4	t^5+t^3	$t^8+t^6+2t^4$	$t^8+t^6+t^4$	$t^7+t^5+t^3$	$t^{10}+t^8+t^6$	$t^9+t^7+2t^5$
$41.$	t^6	t^7+t^5	t^6+t^4	t^6+t^4	t^5	t^8+t^6	t^8+t^6	t^7	$t^{10}+t^8+t^6$	t^9+t^7
2.3	t^5+t^3	t^6	t^5+t^3	t^5	t^4+t^2	$t^7+t^5+t^3$	t^7+t^5	$t^6+t^4+t^2$	t^9	$t^8+t^6+t^4$
31.1	t^5+t^3	$t^6+2t^4+t^2$	t^5+2t^3	t^5+2t^3	t^4	$t^7+2t^5+t^3$	$t^7+2t^5+t^3$	t^6+t^4	$t^9+2t^7+2t^5+t^3$	$t^8+2t^6+2t^4$
1.4	t^4		t^4		t^3+t	t^6	t^6	t^5+t^3		t^7
21.2	t^4+t^2	t^5+t^3	t^4+2t^2	t^4+t^2	t^3	$t^6+2t^4+t^2$	$t^6+2t^4+t^2$	t^5+t^3	$t^8+t^6+t^4$	$t^7+2t^5+2t^3$
3.1^2	t^4	t^5+t^3+t	t^4	t^4	t^3	$t^6+t^4+t^2$	t^6	t^5	$t^8+t^6+t^4$	$t^7+t^5+t^3$
$32.$	t^4	t^5	t^4	t^4+t^2		t^6	t^6+t^4	t^5	$t^8+t^6+t^4$	t^7+t^5
$1^2.3$	t^3		t^3+t		t^2	t^5+t^3	t^5+t^3	t^4		t^6+t^4
2.21	t^3+t	t^4+t^2	t^3	t^3	t^2	t^5+2t^3+t	t^5+t^3	t^4+t^2	t^7+t^5	$t^6+2t^4+t^2$
$2^2.1$	t^3	t^4	t^3	t^3+t		t^5	t^5+t^3	t^4	$t^7+t^5+t^3$	$t^6+t^4+t^2$
1.31	t^2		t^2		t	t^4+t^2	t^4	t^3+t		t^5+t^3
21.1^2	t^2	t^3+t	t^2	t^2		t^4+t^2	t^4+t^2	t^3	$t^6+t^4+t^2$	t^5+2t^3+t
$31^2.$		t^3	t^2	t^2		t^4	t^4		$t^6+t^4+t^2$	t^5
$1^2.21$	t		t			t^3+t	t^3+t	t^2		t^4+2t^2
$21^2.1$		t^2	t	t		t^3	t^3+t		t^5+t^3+t	t^4+t^2
$.5$					t^2			t^4		
1.2^2	1					t^2	t^2	t		t^3
2.1^3		1				t			t^3	t^2
$1^3.2$			1			t^2	t^2			t^3
$2^21.$				1			t^2		t^4+t^2	t^3
$.41$					1			t^2		
1.21^2						1				t
$1^3.1^2$							1			t
$.32$								1		
$21^3.$									1	
$1^2.1^3$										1
$.31^2$										
$1^4.1$										
$.2^21$										
1.1^4										
$.21^3$										
$1^5.$										
$.1^5$										

	$.31^2$	$1^4.1$	$.2^21$	1.1^4	$.21^3$
$5.$	t^{11}	t^{13}	t^{13}	t^{16}	t^{17}
4.1	$t^{10}+t^8+t^6$	$t^{12}+t^{10}+t^8$	$t^{12}+t^{10}+t^8$	$t^{15}+t^{13}+t^{11}+t^9$	$t^{16}+t^{14}+t^{12}+t^{10}$
3.2	$t^9+t^7+2t^5$	$t^{11}+t^9+2t^7$	$t^{11}+t^9+2t^7+t^5$	$t^{14}+t^{12}+2t^{10}+t^8+t^6$	$t^{15}+t^{13}+2t^{11}+2t^9+t^7$
4.1	t^9+t^7	$t^{11}+t^9+t^7$	$t^{11}+t^9$	$t^{14}+t^{12}+t^{10}$	$t^{15}+t^{13}+t^{11}$
$41.$	t^9+t^7	$t^{11}+t^9+t^7$	$t^{11}+t^9$	$t^{14}+t^{12}+t^{10}$	$t^{15}+t^{13}+t^{11}$
2.3	$t^8+t^6+2t^4$	$t^{10}+t^8+t^6$	$t^{10}+t^8+2t^6+t^4$	$t^{13}+t^{11}+t^9+t^7$	$t^{14}+t^{12}+2t^{10}+2t^8+t^6$
31.1	$t^8+2t^6+t^4$	$t^{10}+2t^8+3t^6+t^4$	$t^{10}+2t^8+2t^6$	$t^{13}+2t^{11}+3t^9+2t^7+t^5$	$t^{14}+2t^{12}+3t^{10}+2t^8+t^6$
1.4	$t^7+t^5+t^3$	t^9	$t^9+t^7+t^5$	t^{12}	$t^{13}+t^{11}+t^9+t^7$
21.2	$t^7+2t^5+t^3$	$t^9+2t^7+3t^5+t^3$	$t^9+2t^7+2t^5+t^3$	$t^{12}+2t^{10}+3t^8+2t^6+t^4$	$t^{13}+2t^{11}+3t^9+3t^7+2t^5$
3.1^2	$t^7+t^5+t^3$	$t^9+t^7+t^5$	$t^9+t^7+t^5$	$t^{12}+t^{10}+2t^8+t^6+t^4$	$t^{13}+t^{11}+2t^9+t^7+t^5$
$32.$	t^7	$t^9+t^7+t^5$	t^9+t^7	$t^{12}+t^{10}+t^8$	$t^{13}+t^{11}+t^9$
$1^2.3$	$t^6+t^4+t^2$	$t^8+t^6+t^4$	$t^8+t^6+t^4$	$t^{11}+t^9+t^7$	$t^{12}+t^{10}+2t^8+t^6+t^4$
2.21	$t^6+2t^4+t^2$	$t^8+2t^6+t^4$	$t^8+2t^6+2t^4+t^2$	$t^{11}+2t^9+2t^7+2t^5+t^3$	$t^{12}+2t^{10}+3t^8+3t^6+2t^4$
$2^2.1$	t^6	$t^8+t^6+2t^4$	$t^8+t^6+t^4$	$t^{11}+t^9+2t^7+t^5$	$t^{12}+t^{10}+2t^8+t^6$
1.31	t^5+2t^3+t	t^7+t^5	$t^7+2t^5+2t^3$	$t^{10}+t^8+t^6$	$t^{11}+2t^9+3t^7+2t^5+t^3$
21.1^2	t^5+t^3	$t^7+2t^5+2t^3$	$t^7+2t^5+t^3$	$t^{10}+2t^8+3t^6+2t^4+t^2$	$t^{11}+2t^9+3t^7+2t^5+t^3$
$31^2.$	t^5	$t^7+t^5+t^3$	t^7	$t^{10}+t^8+t^6$	$t^{11}+t^9+t^7$
$1^2.21$	t^4+t^2	$t^6+2t^4+t^2$	$t^6+2t^4+t^2$	$t^9+2t^7+2t^5+t^3$	$t^{10}+2t^8+3t^6+2t^4+t^2$
$21^2.1$	t^4	$t^6+2t^4+2t^2$	t^6+t^4	$t^9+2t^7+2t^5+t^3$	$t^{10}+2t^8+2t^6+t^4$
$.5$	t^6		t^8		t^{12}
1.2^2	t^3	t^5	t^5+t^3+t	t^8+t^4	$t^9+t^7+2t^5+t^3$
2.1^3	t^2	t^4	t^4	$t^7+t^5+t^3+t$	$t^8+t^6+t^4+t^2$
$1^3.2$	t^3	t^5+t^3+t	t^5	$t^8+t^6+t^4$	$t^9+t^7+t^5+t^3$
$2^21.$		t^5+t^3	t^5	t^8+t^6	t^9+t^7
$.41$	t^4+t^2		t^6+t^4		$t^{10}+t^8+t^6$
1.21^2	t	t^3	t^3+t	$t^6+t^4+t^2$	$t^7+2t^5+2t^3+t$
$1^3.1^2$		t^3+t	t^3	$t^6+t^4+t^2$	$t^7+t^5+t^3$
$.32$	t^2		t^4+t^2		$t^8+t^6+t^4$
$21^3.$		t		t^4	t^5
$1^2.1^3$		t^2	t^2	t^5+t^3+t	$t^6+t^4+t^2$
$.31^2$	1		t^2		$t^6+t^4+t^2$
$1^4.1$		1		t^3	t^4
$.2^21$			1		t^4+t^2
1.1^4				1	t
$.21^3$					1
$1^5.$					
$.1^5$					

	$1^5.$	$.1^5$
$5.$	t^{20}	t^{25}
4.1	$t^{19}+t^{17}+t^{15}+t^{13}$	$t^{24}+t^{22}+t^{20}+t^{18}+t^{16}$
3.2	$t^{18}+t^{16}+2t^{14}+t^{12}+t^{10}$	$t^{23}+t^{21}+2t^{19}+2t^{17}+2t^{15}+t^{13}+t^{11}$
$41.$	$t^{18}+t^{16}+t^{14}+t^{12}$	$t^{23}+t^{21}+t^{19}+t^{17}$
2.3	$t^{17}+t^{15}+t^{13}+t^{11}$	$t^{22}+t^{20}+2t^{18}+2t^{16}+2t^{14}+t^{12}+t^{10}$
31.1	$t^{17}+2t^{15}+3t^{13}+3t^{11}+2t^9+t^7$	$t^{22}+2t^{20}+3t^{18}+3t^{16}+3t^{14}+2t^{12}+t^{10}$
1.4	t^{16}	$t^{21}+t^{19}+t^{17}+t^{15}+t^{13}$
21.2	$t^{16}+2t^{14}+3t^{12}+3t^{10}+2t^8+t^6$	$t^{21}+2t^{19}+3t^{17}+4t^{15}+4t^{13}+3t^{11}+2t^9+t^7$
3.1^2	$t^{16}+t^{14}+2t^{12}+t^{10}+t^8$	$t^{21}+t^{19}+2t^{17}+2t^{15}+2t^{13}+t^{11}+t^9$
$32.$	$t^{16}+t^{14}+t^{12}+t^{10}+t^8$	$t^{21}+t^{19}+t^{17}+t^{15}+t^{13}$
$1^2.3$	$t^{15}+t^{13}+t^{11}+t^9$	$t^{20}+t^{18}+2t^{16}+2t^{14}+2t^{12}+t^{10}+t^8$
2.21	$t^{15}+2t^{13}+2t^{11}+2t^9+t^7$	$t^{20}+2t^{18}+3t^{16}+4t^{14}+4t^{12}+3t^{10}+2t^8+t^6$
$2^2.1$	$t^{15}+t^{13}+2t^{11}+2t^9+t^7+t^5$	$t^{20}+t^{18}+2t^{16}+2t^{14}+2t^{12}+t^{10}+t^8$
1.31	$t^{14}+t^{12}+t^{10}$	$t^{19}+2t^{17}+3t^{15}+3t^{13}+3t^{11}+2t^9+t^7$
21.1^2	$t^{14}+2t^{12}+3t^{10}+3t^8+2t^6+t^4$	$t^{19}+2t^{17}+3t^{15}+4t^{13}+4t^{11}+3t^9+2t^7+t^5$
$31^2.$	$t^{14}+t^{12}+2t^{10}+t^8+t^6$	$t^{19}+t^{17}+2t^{15}+t^{13}+t^{11}$
$1^2.21$	$t^{13}+2t^{11}+2t^9+2t^7+t^5$	$t^{18}+2t^{16}+3t^{14}+4t^{12}+4t^{10}+3t^8+2t^6+t^4$
$21^2.1$	$t^{13}+2t^{11}+3t^9+3t^7+2t^5+t^3$	$t^{18}+2t^{16}+3t^{14}+3t^{12}+3t^{10}+2t^8+t^6$
$.5$		t^{20}
1.2^2	$t^{12}+t^8$	$t^{17}+t^{15}+2t^{13}+2t^{11}+2t^9+t^7+t^5$
2.1^3	$t^{11}+t^9+t^7+t^5$	$t^{16}+t^{14}+2t^{12}+2t^{10}+2t^8+t^6+t^4$
$1^3.2$	$t^{12}+t^{10}+2t^8+t^6+t^4$	$t^{17}+t^{15}+2t^{13}+2t^{11}+2t^9+t^7+t^5$
$2^21.$	$t^{12}+t^{10}+t^8+t^6+t^4$	$t^{17}+t^{15}+t^{13}+t^{11}+t^9$
$.41$		$t^{18}+t^{16}+t^{14}+t^{12}$
1.21^2	$t^{10}+t^8+t^6$	$t^{15}+2t^{13}+3t^{11}+3t^9+3t^7+2t^5+t^3$
$1^3.1^2$	$t^{10}+t^8+2t^6+t^4+t^2$	$t^{15}+t^{13}+2t^{11}+2t^9+2t^7+t^5+t^3$
$.32$		$t^{16}+t^{14}+t^{12}+t^{10}+t^8$
$21^3.$	$t^8+t^6+t^4+t^2$	$t^{13}+t^{11}+t^9+t^7$
$1^2.1^3$	$t^9+t^7+t^5+t^3$	$t^{14}+t^{12}+2t^{10}+2t^8+2t^6+t^4+t^2$
$.31^2$		$t^{14}+t^{12}+2t^{10}+t^8+t^6$
$1^4.1$	$t^7+t^5+t^3+t$	$t^{12}+t^{10}+t^8+t^6+t^4$
$.2^21$		$t^{12}+t^{10}+t^8+t^6+t^4$
1.1^4	t^4	$t^9+t^7+t^5+t^3+t$
$.21^3$		$t^8+t^6+t^4+t^2$
$1^5.$	1	t^5
$.1^5$		1

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